Power Gain Analysis and Control of Nonlinear Systems

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Declaration

These doctoral studies were carried out with supervision from Dr. Matthew R. James of the Department of Engineering, Faculty of Engineering and Information Technology, The Australian National University. Advisory support was provided by Dr. Michael Green, also of the Department of Engineering, Faculty of Engineering and Information Technology, and Professor Brian Anderson of the Department of Systems Engineering, Research School of Information Sciences and Engineering, The Australian National University.

The work contained in this thesis, except where explicitly stated, is original research whose major portion has been done by the author. The work has not been submitted for a degree at any other university or institution.

Most of the research contained in this thesis has been published or submitted to journals and conferences as listed below.

Journal Papers:

[J1] P.M. Dower, M.R. James, *"Dissipativity and Nonlinear Systems with Finite Power Gain"*, *to appear, Int. J. Robust and Nonlinear Control*, 1998.

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[C1] P.M. Dower, M.R. James, *"Finite Power Gain Analysis of Nonlinear Systems"*, *IEAust Control 95 Conference, Melbourne, Australia*, vol 1, pp.79-83, 1995.

[C2] P.M. Dower, M.R. James, *"State Feedback Finite Power Gain Control for Nonlinear Systems"*, *1996 IEEE Conference on Decision and Control, Kobe, Japan*, 1996.

[C3] P.M. Dower, P.G. Dupuis, M.R. James, *"Numerical Schemes for the Finite Power Gain Problem"*, *1997 IEEE Conference on Decision and Control, San Diego Calif., USA*, 1997.

[C4] P.M. Dower, M.R. James, *"Power Gain Control for Linear Systems with Actuator Nonlinearities"*, *IEAust Control 97 Conference, Sydney, Australia*, 1997.

[C5] P.M. Dower, M.R. James, *"Stability of Nonlinear Systems with Worst Case Power Gain Disturbances"*, *Submitted to 1998 IEEE Conference on Decision and Control, Tampa Florida, USA*, 1998.

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Abstract

In this thesis, a theory of power gain analysis is developed for a large class of nonlinear dynamical systems. This theory provides techniques for analysing systems which exhibit internal power generation (such as limit cycle systems) which are generally not treatable using standard $\mathcal{L}_2$-gain analysis (or $\mathcal{H}_\infty$) techniques.

In the absence of disturbances, the stability of nonlinear systems with power gain is investigated. A sufficient stability based condition is developed for a system to exhibit power gain. A natural generalization of the concept of dissipativity, called power dissipativity, is developed for nonlinear systems with power gain, requiring the existence of a power bias / storage function solution pair of a dissipation inequality. A number of candidate power bias / storage function pairs are then proposed, including revised definitions of available storage (an infinite horizon available storage and a super available storage) and required supply. The attendant partial differential equations are also presented. The analysis then proceeds with these revised definitions of available storage and required supply in order to develop an analogous ordering of storage functions for power dissipative systems. The stability of nonlinear systems in the presence of the worst case disturbance is then analysed, uncovering a wealth of new system behaviour, including the bifurcation of equilibria.

The scope of this thesis is then broadened to include the application of power gain analysis to the problem of state feedback controller synthesis. Particular attention is paid to the formulation of an optimal control problem which, if solvable, provides the recipe for a state feedback controller which minimizes the limit cycle behaviour of the nonlinear system in closed loop. This theory is then applied to a class of linear systems with actuator nonlinearities.

The remainder of this thesis is then directed towards the implementation details of power gain analysis techniques.

Verification of the power gain property requires the existence of a power bias / storage function pair which satisfies the dissipation inequality. Although such pairs are by themselves difficult to find, the analysis provides us with a number of useful candidate pairs, defined as variational problems. Since explicit solutions of such variational problems are rare, the important issue of computing approximations for the candidate power bias / storage function pairs is addressed. The finite difference methods presented rely heavily on similar existing methods developed for stochastic control problems.

Finally, the richness of behaviour of explicit nonlinear systems with power gain is investigated, revealing a real departure from the known behaviour of nonlinear systems with finite $\mathcal{H}_\infty$ norm. Applying numerical techniques, the candidate power bias / storage function pairs which arise in the theory of power gain analysis are computed. The stability of nonlinear system with power gain is investigated in the absence of disturbances, and in the presence of the worst case power gain disturbance. Systems such as simple as scalar linear systems with output saturation are investigated, along with two dimensional limit cycles systems, and three dimensional chaotic systems.

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Notation and Abbreviations

Notation

This section describes the notation that is used throughout this thesis.

$\mathbf{R}$ denotes the real line, whilst $\mathbf{R}^n$ denotes n-dimensional euclidean space. $|x|$ refers to the n-dimensional euclidean norm of x. For $x,\ y \in \mathbf{R}^n$, $x \cdot y$ is the euclidean inner product of x and y. $B[x_0;r]$ ($B(x_0;r)$) refers to the closed (open) ball of center x_0 and radius r in $\mathbf{R}^n$.

$\mathbf{R}^{n\times m}$ denotes the space of $n \times m$ matrices with real entries. If $M \in \mathbf{R}^{n\times m}$, then M_{ij} denotes the entries of matrix M corresponding to the i^{th} row and j^{th} column. M' refers to the transpose. If M is stable, then all eigenvalues of M lie in the open left half of the complex plane. Unless otherwise stated, $|M|$ refers to the matrix supremum norm of M. That is, the $|M|$ is the maximum singular value of M.

A function $V : \mathbf{R}^n \to \mathbf{R}$ maps values in $\mathbf{R}^n$ to values in $\mathbf{R}$. If $V \in C^1(\mathbf{R}^n, \mathbf{R})$ (or just C^1), then V is differentiable. $\nabla_x V(x)$ refers to the gradient of V, $\left[\frac{\partial V}{\partial x_1}|\frac{\partial V}{\partial x_2}|\cdots|\frac{\partial V}{\partial x_n}\right]$, where $x = [x_1|x_2|\cdots|x_n]'$. This may be abbreviated to V_x. The argmin of V is a set given by $\operatorname{argmin}_{x\in\mathbf{R}^n}\{V(x)\} = \{x \in \mathbf{R}^n : V(x) \leq V(y) \text{ forall } y \in \mathbf{R}^n\}$. Argmax is defined similarly. The lower semicontinuous envelope of V is written as V_*, where $V_*(x) = \liminf_{z\to x}\{V(z)\}$. The normalization of V is denoted by $\overline{V}(x) = V(x) - \inf_{x\in\mathbf{R}^n} V(x)$.

A signal $v : \mathbf{R} \to \mathbf{R}^p$ is a function which maps times to vectors. $v \in \mathcal{L}_2([0,T], \mathbf{R}^p)$ if

$$\int_0^T |v(t)|^2 \, dt \;<\; \infty.$$

Where the dimension of $v \in \mathbf{R}^p$ is clear from the context, $v \in \mathcal{L}_2([0,T], \mathbf{R}^p)$ may be abbreviated to $v \in \mathcal{L}_2[0,T]$. If $v \in \mathcal{L}_2[0,T]$, the $\mathcal{L}_2$-norm of v is given by

$$\|v\|_{\mathcal{L}_2[0,T]} = \sqrt{\int_0^T |v(t)|^2 \, dt}.$$

The union of $\mathcal{L}_2([0,T], \mathbf{R}^p)$ spaces for all $T > 0$ is denoted by

$$\mathcal{L}_{2e}(\mathbf{R}^p) = \cup_{T>0} \{\mathcal{L}_2([0,T], \mathbf{R}^p)\},$$

which again may be abbreviated to $\mathcal{L}_{2e}[0,T]$. Signal $v \in \mathcal{L}_2([0,\infty), \mathbf{R}^p)$ if

$$\int_0^\infty |v(t)|^2 \, dt < \infty.$$

$v \in \mathcal{L}_2([0,\infty), \mathbf{R}^p)$ may be abbreviated to $v \in \mathcal{L}_2$. If $v \in \mathcal{L}_2$, the $\mathcal{L}_2$-norm of v is given by

$$\|v\|_{\mathcal{L}_2} = \sqrt{\int_0^T |v(t)|^2 \, dt}.$$

Signal $v \in \mathcal{FP}(\mathbf{R}^p)$ if

$$\limsup_{T \to \infty} \left\{ \frac{1}{T} \int_0^T |v(t)|^2 \, dt \right\} < \infty.$$

Where the dimension of $v \in \mathcal{FP}(\mathbf{R}^p)$ is clear from the context, this may be written as $v \in \mathcal{FP}$. If $v \in \mathcal{FP}$, the $\mathcal{FP}$-norm of v is given by

$$\|v\|_{\mathcal{FP}} = \sqrt{\limsup_{T \to \infty} \left\{ \frac{1}{T} \int_0^T |v(t)|^2 \, dt \right\}}.$$

The minimum of two real valued quantities a and b is denoted by $a \wedge b$, whilst the maximum is denoted by $a \vee b$. Indicator functions are denoted by

$$\chi_b = \begin{cases} 1 & \text{if } \; b \text{ evaluates to } \mathit{TRUE}, \\ 0 & \text{if } \; b \text{ evaluates to } \mathit{FALSE}. \end{cases}$$

Abbreviations

The following abbreviations are used in this thesis:

ARE	Algebraic Riccati Equation
DI	Dissipation Inequality
DPE	Dynamic Programming Equation
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
PDI	Partial Differential Inequality
RDE	Riccati Differential Equation
HJB	Hamilton-Jacobi-Bellman
HJI	Hamilton-Jacobi-Isaacs

Chapter 1

Introduction

1.1 Analysis of Nonlinear Systems with Energy Gain

Over the last decade, significant breakthroughs have been made in generalizing concepts of linear $\mathcal{H}_\infty$ control [44, 13] to include nonlinear systems [38, 23, 1, 26]. The fundamental step in this generalization has been the interpretion of the frequency domain definition of the $\mathcal{H}_\infty$ norm of a system in terms of time domain energy ($\mathcal{L}_2$-) gain. With this interpretation has come renewed interest in the fields of $\mathcal{L}_2$-gain analysis [39], and more fundamentally, dissipative systems [42, 21, 22].

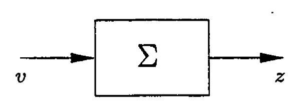


Figure 1.1: Input/Output View of System Σ

$\mathcal{L}_2$-gain analysis is concerned with the input / output analysis of systems with finite $\mathcal{L}_2$-gain. That is, systems which satisfy the inequality

$$\int_0^T |z(s)|^2\, ds \;\; \le \;\; \gamma^2 \int_0^T |v(s)|^2\, ds + \beta(x),$$

for all inputs v, times T, and states x, where the finite gain γ is fixed, and z represents the output. With appropriate detectability properties, the $\mathcal{L}_2$-gain inequality implies that the system must be asymptotically stable in the absence of inputs.

Upon inspection of the $\mathcal{L}_2$-gain inequality, it is not clear a priori how such an input / output property can be established. However, since the inequality must hold for all inputs, the problem of testing for $\mathcal{L}_2$-gain can be reformulated in terms of a finite horizon calculus of variations problem. By application of standard dynamic programming techniques [28, 17], the finite horizon value function

$$V(x,T) = \sup_{v\in\mathcal{L}_2} \left\{ \int_0^T \left[|z(s)|^2 - \gamma^2|v(s)|^2\right] \, ds : x(0) = x \right\}$$

may be computed by means of an associated nonstationary partial differential equation. Existence of a finite nonnegative function $\beta(x)$ such that

$$V(x,T) \leq \beta(x)$$

holds for all states x and times T clearly implies that the $\mathcal{L}_2$-gain inequality holds.

The finite horizon value function $V(x,T)$ can be interpreted as the most energy that can be retrieved from a system, starting in state x, in a finite time interval $[0,T]$. Not suprisingly, it follows that $V(x,T)$ is nondecreasing in T. Hence, as the finite horizon value function $V(x,T)$ is uniformly bounded above by $\beta(x)$, it is clear that an infinite horizon limit of the finite horizon value function $V(x,T)$ must exist and also be bounded above by $\beta(x)$. Analysis of these infinite horizon functions is the essence of dissipative systems theory.

1.2 Energy Dissipative Systems

On an abstract level, dissipative systems theory is concerned with the relationship between energy flow into and out from systems, and internal stability. One of the properties of dissipative systems is their ability to store energy. This stored energy, or *storage*, represents a finite reservoir of energy to which energy may be supplied or from which energy may be withdrawn by the application of inputs or disturbances to the system. The level of storage is represented by a *storage function*, which is simply a nonnegative scalar function of the state of the system. In the absence of disturbances, finiteness of the initial storage ensures that as energy is delivered to the external environment by way of the output and by way of energy *dissipation*, the storage must eventually decay to zero. With suitable detectability properties, this implies that the state of the system must itself decay to the origin. In this way, dissipative systems can be shown to be *internally stable* [42, 21, 22].

Formally, a system is energy dissipative (or just simply *dissipative*) with gain γ if a finite nonnegative *storage function* can be found which satisfies the *dissipation* *inequality*

$$V(x) + \int_0^T r(v(s), z(s))\, ds \;\geq\; V(x(T))$$

for all disturbances v, all times T, and all states x, where r is the *supply rate.* Since V is a measure of the stored energy and r is a measure of the rate of energy supply to the system, clearly the dissipation inequality is an energy balance relationship. The notion of energy dissipation is embodied by the fact that this balance holds with inequality rather than equality. In applying dissipative systems theory to $\mathcal{L}_2$-gain analysis, the supply rate is chosen naturally to be $r(v,z) = \gamma^2|v|^2 - |z|^2$.

Note that for dissipativity to be of use, it is imperative that there is a link between dissipativity and the $\mathcal{L}_2$-gain property. Comparing the $\mathcal{L}_2$-gain inequality and the dissipation inequality, the existence of a nonnegative storage function implies immediately that the $\mathcal{L}_2$-gain inequality holds. Furthermore, using a concept known as *available storage*, it can be shown that systems with finite $\mathcal{L}_2$-gain must be dissipative. Assumptions regarding the detectability of a system also admits treatment of any storage function as a Lyapunov function. Application of LaSalle's Principle then implies asymptotic stability for dissipative systems. Hence, dissipativity plays an important role in ensuring that a system with $\mathcal{L}_2$-gain is internally stable.

Although a storage function V characterizes the dissipativeness of a system, it is generally non-unique. Without a recipe, the problem of finding a candidate storage function can be difficult. However, this problem can be avoided by defining the concepts of *available storage* and *required supply* [42]. As the name suggests, the available storage $V_a(x)$ of a system is defined uniquely to be the most energy retrievable from that system, starting in a particular state, by the application of any disturbance over any time horizon. That is,

$$V_a(x) \;=\; \sup_{T\geq 0}\, \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[|z(s)|^2 - \gamma^2|v(s)|^2\right]\, ds : x(0) = x \right\}.$$

Not suprisingly, the available storage is the limiting function as $T \to \infty$ of the finite horizon value function $V(x,T)$. That is,

$$V_a(x) \;=\; \sup_{T\geq 0}\{V(x,T)\} \;=\; \lim_{T\to\infty}\{V(x,T)\}.$$

The required supply $V_r(x)$ is the least energy required to achieve a final state x, starting from a state of zero storage, over any time horizon. That is,

$$V_r(x) = \inf_{T\geq 0} \inf_{v\in\mathcal{L}_2[-T,0]} \left\{ \int_{-T}^{0} \left[\gamma^2|v(s)|^2 - |z(s)|^2\right] \, ds : x(-T) = 0, \; x(0) = x \right\}.$$

As with the available storage, the required supply can be shown to be the infinite horizon limit of a finite horizon value function.

The available storage and required supply necessarily satisfy the dissipation inequality. Hence, finiteness of either immediately implies dissipativeness. So, the variational definitions of available storage and required supply naturally give rise to a prescription for dissipativeness: compute either $V_a(x)$ or $V_r(x)$, test for finiteness.

Although the available storage and required supply have explicit definitions, these definitions are still variational, and as such, difficult to compute. However, by considering a differential form of the dissipation inequality, a verification result for dissipativeness follows. That is, the existence of a solution V of a corresponding partial differential inequality

$$H(x, \nabla V(x)) \leq 0$$

implies (by integration) that V is a solution of the dissipation inequality. Furthermore, the available storage and required supply play a special role in the solution of the corresponding partial differential equation: $V_a(x)$ is the *stabilizing* solution, whilst $V_r(x)$ is the *antistabilizing* solution. For linear systems, this partial differential equation reduces to the algebraic Riccati equation, for which solutions techniques exist. However, for nonlinear systems, numerical approximation techniques are often required in order to compute solutions of the partial differential equation.

In the integral form of the dissipation inequality, the stabilizing and antistabilizing properties of the available storage and required supply are manifested in the way that they delimit all possible storage functions for a system. That is, any storage function V necessarily satifies the inequality

$$V_a(x) \leq V(x) \leq V_r(x).$$

Hence, the available storage is the minimal storage function, whilst the required supply is the maximal storage function.

1.3 Analysis of Nonlinear Systems with Power Gain

Asymptotic stability of detectable systems with the $\mathcal{L}_2$-gain property immediately implies that systems with nonzero steady state disturbance free behaviour cannot be treated directly with $\mathcal{L}_2$-gain analysis or energy dissipative techniques. Hence, a large class of systems with very interesting dynamics, including for example limit cycle and chaotic systems, are not amenable to analysis using these existing techniques [38, 39].

Although such systems do not exhibit $\mathcal{L}_2$-gain, many exhibit (finite) *power* ($\mathcal{FP}$-) gain. The power gain property in itself represents a very simple generalization of the $\mathcal{L}_2$-gain inequality. The only difference between the two properties is an additional linear term in T on the RHS of the *power gain inequality.* That is, a system exhibits $\mathcal{FP}$-gain $\leq \gamma$ if there exists two nonnegative functions λ and β which satisfy the inequality

$$\int_0^T |z(s)|^2 \, ds \quad \leq \quad \gamma^2 \int_0^T |v(s)|^2 \, ds + \lambda(x)T + \beta(x)$$

for all disturbances v, all times T, and all states x. The coefficient of the linear T term, $\lambda(x)$, is referred to as the *power bias* of the system. By inspection of the power gain inequality, this power bias term facilitates the presences of nonzero steady state outputs in the absence of disturbances. That is, the power delivered by limit cycle behaviour for example can be accounted for with the $\lambda(x)T$ term.

Under an assumption of exponential stability outside a compact set, a large class of nonlinear systems (including many limit cycle systems) can be shown to exhibit $\mathcal{FP}$-gain. Conversely, under suitable detectability assumptions, systems with $\mathcal{FP}$-gain can be shown to be stable in the sense that trajectories unperturbed by disturbances tend to a compact set. In the case where the power bias is zero ($\lambda(x) \equiv 0$), these results naturally recover the corresponding standard $\mathcal{L}_2$-gain analysis results.

As with the $\mathcal{L}_2$-gain property, it is possible to reformulate the power gain inequality in terms of the finite horizon value function $V(x,T)$. Then, the power gain inequality becomes

$$V(x,T) \quad \leq \quad \lambda(x)T + \beta(x).$$

A substantial difference in the power gain case arises from the fact that this value function $V(x,T)$ need no longer be uniformly bounded above for all T. Indeed, it is follows from the above inequality that systems with power gain may have a finite horizon value function $V(x,T)$ which grows linearly (or sublinearly) with T. However, since $V(x,T)$ is representative of the most energy that may be retrieved from a system, it follows naturally that a concept of the most power deliverable by a system may be developed. This maximal power generation, called the *available power* λ_a, is defined simply as the infinite horizon linear growth of the energy retrieved. That is,

$$\lambda_a(x) = \limsup_{T\to\infty} \left\{ \frac{V(x,T)}{T} \right\}.$$

By applying this definition of available power to the power gain inequality, it is apparent that the available power represents a lower bound for any power bias for the system. Furthermore, under appropriate reachability assumptions, the power gain is naturally independent of the initial state x. Consequently, both the available power and any admissible power bias are usually considered to be independent of x.

For systems with $\mathcal{L}_2$-gain, the available power must be zero. This follows from the fact that such systems do not have the ability to generate energy internally. However, for systems which do not possess the $\mathcal{L}_2$-gain property, the available power is typically a nonnegative function of the gain γ. Furthermore, the "worst case" disturbance which excites the maximum power generation of a system (and thereby defines the available power) depends explicitly on the rate of change of the available power with respect to gain. That is,

$$\|v_\gamma^*\|_{\mathcal{FP}} = \sqrt{\frac{-1}{2\gamma}\left(\frac{d\lambda_a^\gamma}{d\gamma}\right)},$$

where v_γ^* is this worst case disturbance for gain γ, and $\|\cdot\|_{\mathcal{FP}}$ is a power seminorm. By inspection, the worst case disturbance v_γ^* can only have nonzero power if the available power is decreasing with gain. This illustrates further that the available power is indeed a property of systems for which power signals, rather than energy signals, are important.

1.4 Power Dissipative Systems

An important property of systems with nonzero available power is the ability of such systems to generate power in the absence of disturbances. Clearly such systems cannot be energy dissipative, as this internal power generation would imply infinite energy storage. Hence, a generalization of energy dissipativity is required in order to cope with systems with nonzero available power.

For systems with power gain, the power bias λ can be regarded as an approximation of the internal power generation of a system. With this in mind, a simple generation of dissipativity is obtained by adding the internal power generation to the supply rate, yielding $r(v, z) = \gamma^2|v|^2 - |z|^2 + \lambda$. Hence, a system is power dissipative if there exists a finite nonnegative power bias / storage function pair such that

$$V(x) + \int_0^T \left[\gamma^2|v(s)|^2 - |z(s)|^2 + \lambda\right] \, ds \quad \geq \quad V(x(T))$$

for all disturbances v, times T, and states x. It is not difficult to show that power dissipativity and power gain are equivalent concepts. However, a notable departure from energy dissipative systems theory is that a power bias / storage function *pair* must be found in order verify power dissipativeness. Since such pairs are nonunique, the approach is again to propose and verify candidate solutions to the dissipation inequality.

An obvious choice for a candidate power bias / storage function pair is the available power / available storage pair. However, the energy dissipative definition of available storage is not suitable, since it is infinite for nonzero available power. Hence, a generalization of available storage is required. One possibility is the *super available storage*,

$$V_a(x) \quad = \quad \sup_{T \geq 0} \{V(x,T) - \lambda_a T\},$$

where $V(x,T)$ is the same finite horizon value function defined in $\mathcal{L}_2$-gain analysis. Although the pair (λ_a, V_a) does satisfy the dissipation inequality for power dissipative systems, it must be stressed that the super available storage is not the only possible generalization. To see this, recall that in $\mathcal{L}_2$-gain analysis, $\lambda_a = 0$. Hence, monotonicity of $V(x,T)$ guarantees that replacing the supremum with a limit (as $T \to \infty$) in the definition of available storage for energy dissipative systems has no effect. However, for $\lambda_a > 0$, the monotonicity of $V(x,T)$ is lost, giving rise to an alternative generalization called the *infinite horizon available storage*,

$$V_b(x) \quad = \quad \limsup_{T \to \infty} \{V(x,T) - \lambda_a T\}.$$

Note that the pair (λ_a, V_b) also satisfies the dissipation inequality.

Since both $V_a(x)$ and $V_b(x)$ are valid generalizations of available storage, the two are distinguished by noting that $V_a(x) \geq V_b(x)$ (hence the name "super" available storage).

Importantly, the super available storage may be interpreted as the value function of an optimal stopping problem [3]. Applying the results of [3, 37], this implies that the super available storage is a solution of the variational inequality

$$\max\left(-\lambda_a + H(x, \nabla V_a(x)), -V_a(x)\right) = 0.$$

In contrast, it can be concluded that the infinite horizon available storage is a solution of the PDE

$$-\lambda_a + H(x, \nabla V_b(x)) = 0.$$

Clearly for $\lambda_a = 0$, both differential equations reduce to the well known Hamilton-Jacobi-Bellman equation associated with $\mathcal{L}_2$-gain analysis [38].

Note that with regard to utility, we find that the super available storage is useful for analysing stability in the absence of disturbances, whilst the infinite horizon available storage is useful for analysing stability in the presence of the worst case power gain disturbance.

In keeping with the generalizations thus far, the required supply for system with power gain may also be defined. Given that the required supply was defined in [42] as the least energy required to move from a state of minimum storage to a state x, we define a generalization of the required supply to be the infinite horizon fixed initial state required supply,

$$V_{br}^f(\xi, x) = \liminf_{T\to\infty} \inf_{v\in\mathcal{L}_2[-T,0]} \left\{ \int_{-T}^{0} \left[\gamma^2|v(s)|^2 - |z(s)|^2 + \lambda_a\right] ds : x(-T) = \xi, x(0) = x \right\},$$

where ξ is chosen to be a minimizing state of the infinite horizon available storage $V_b(x)$.

The definition of required supply along with that of the infinite horizon available storage leads to one of the fundamental differences between $\mathcal{L}_2$-gain analysis and power gain analysis. One of the features of systems with $\mathcal{L}_2$-gain is that the equilibrium of the system remains unchanged regardless of whether the system is in the presence of the worst case disturbance or no disturbance. However, systems with power gain can exhibit quite difference behaviour from when the system is free of disturbances to when the worst case power gain disturbance is applied. Typical behaviour includes the shifting of equilibria, the bifurcation of equilbria, the change in shape and size of limit cycles, etc. This change in behaviour is reflected in the fact that minimum of $V_b(x)$ no longer corresponds to the steady state worst case dynamics of the system. However, by defining the function

$$W(x) = V_{br}^f(\xi, x) - V_b(x),$$

where $\xi \in \mathrm{argmin}_{x \in \mathbf{R}^n}\{V_b(x)\}$, we find that $W(x)$ decreases along forward time worst case trajectories corresponding to the stabilizing PDE solution $V_b(x)$.

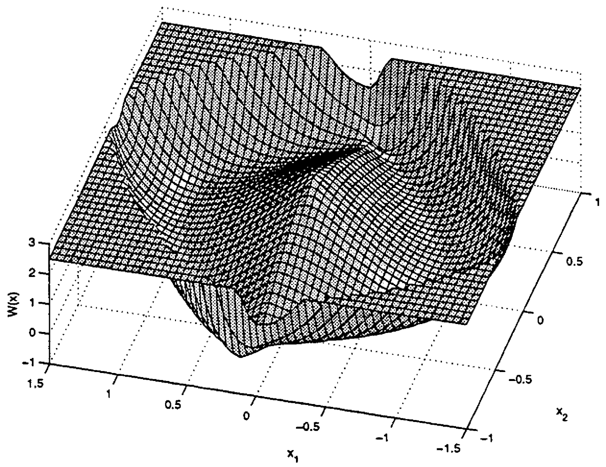


Figure 1.2: $W(x)$ Approximation for a Limit Cycle System

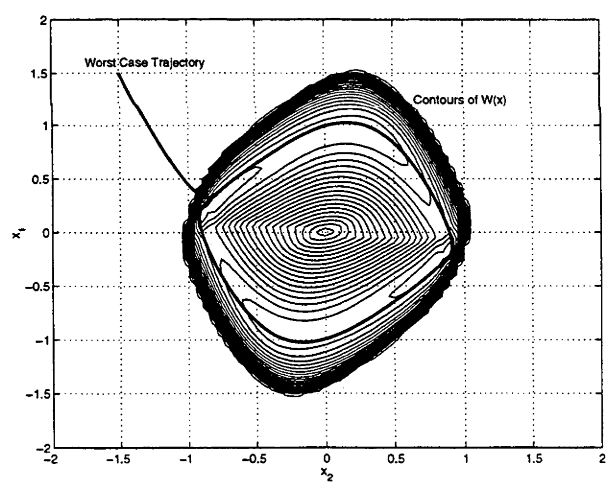


Figure 1.3: A Worst Case Trajectory and the Contours of a $W(x)$ Approximation

Furthermore, $W(x)$ also decreases along reverse time worst case trajectories corresponding to the antistabilizing PDE solution $V_{br}^{f}(\xi, x)$. Hence, the argmin of $W(x)$ forms an invariant set to which both forward and reverse time worst case trajectories are attracted. Consequently, worst case trajectories tend to states for which $V_b(x)$ and $V_{br}^{f}(\xi, x)$ have equal gradients (with respect to x, if they exist), rather than to any minima of either function. Note that in the $\mathcal{L}_2$-gain analysis case it is the zero available power that guarantees that the states of common gradient and common minima coincide at the origin.

1.5 Optimal State Feedback Power Gain Control

A closed loop system consisting of a plant G and a controller K is an example of a system with an input and output. Consequently, the input / output system (G, K) may be analysed using the power gain analysis techniques outlined thus far.

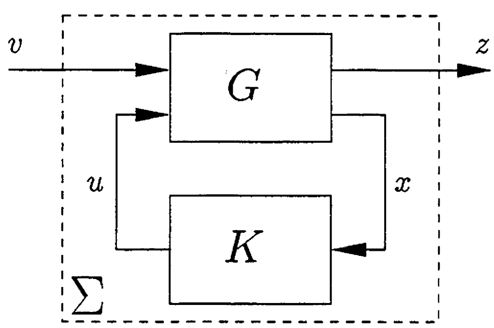


Figure 1.4: Closed Loop System $\Sigma = (G, K)$

Since the controller K modifies the behaviour of system (G, K), it is reasonable to expect that different choices of controller K may yield different levels of internal power generation, and hence different values of available power. Thus, we may associate with system (G, K) the available power $\lambda_{a,K}$, which is dependent on the choice of controller K. Furthermore, since the available power is always bounded below by zero,

$$0 \leq \min_K \{\lambda_{a,K}\} \leq \lambda_{a,K},$$

for any controller K. Hence, it is feasible to define an *optimal state feedback power gain control problem* which involves finding an optimal controller K^* such that

$$\lambda_{a,K^*} = \min_K \{\lambda_{a,K}\}.$$

As the available power $\lambda_{a,K}$ may be interpreted as a measure of the worst case nonzero steady state behaviour of system (G, K), it is clear that the optimal state feedback controller is desirable in that it seeks to minimize the closed loop "oscillatory" behaviour of the system. Indeed, if the nonlinear state feedback $\mathcal{H}_\infty$-control problem is solvable, then the optimal state feedback power gain control problem will yield the optimal state feedback $\mathcal{H}_\infty$ controller K^* with $\lambda_{a,K^*} = 0$ and the closed loop (G, K^*) asymptotically stable.

By applying existing deterministic techniques arising from infinite horizon risk sensitive control [16], a solution of the optimal state feedback power gain control problem can be found by computing the solution of a corresponding Hamilton-Jacobi-Isaacs equation [16].

Using this method of solution, the synthesis of optimal state feedback power gain controllers for a class of linear plants with actuator nonlinearities is considered. Typically, we find that the optimal controller attempts to invert the actuator nonlinearity in the spirit of Inverse Control [40].

1.6 Thesis Outline

The following list summarizes the chapter by chapter content of this thesis:

Chapter 2 The theory of power gain analysis is developed for a large class of nonlinear dynamical systems.

Chapter 3 The analysis ideas of Chapter 2 are broadened further to investigate the applicability of power gain analysis to the problem of state feedback controller synthesis.

Chapter 4 The issue of computing appoximations for the candidate power bias / storage function pairs proposed in Chapter 2 is addressed.

Chapter 5 The richness of behaviour of explicit nonlinear systems with power gain is investigated, revealing a real departure from the known behaviour of nonlinear systems with finite $\mathcal{L}_2$-gain.

Chapter 6 The contributions of this thesis are summarized. Further work in this new research area is detailed.

1.7 Summary of Contributions

The principal contributions of this thesis are listed below.

- Generalization of the $\mathcal{L}_2$-gain inequality to the power gain inequality.
- Analysis of the stability of nonlinear systems with power gain, in the absence of disturbances.
- Determination of a large class of nonlinear systems which exhibit power gain.

- Interpretation of the power gain inequality using a finite horizon calculus of variations problem.
- The concept of available power for nonlinear systems with power gain.
- Generalization of energy dissipativity to power dissipativity, thereby including nonlinear systems with power gain.
- Concepts of available storage and required supply for nonlinear systems with power gain.
- Analysis of the stability of nonlinear systems with power gain, in the presence of the worst case disturbance.
- Formulation of the optimal state feedback power gain control problem, with application to linear systems with actuator nonlinearities.
- All explicit computations for nonlinear systems with power gain.

Chapter 2

Continuous Time Power Gain Analysis

2.1 Introduction

Many nonlinear systems, such as limit cycles systems, exhibit internal power generation which is manifested in the persistence of outputs in the absence of disturbances. Since such systems produce an infinite amount of energy, treatment using standard energy gain analysis techniques is not possible.

In order to overcome this limitation in the theory, we propose a generalization of energy gain to include systems with power gain. By developing concepts such as power dissipativity, a theory of power gain analysis is established.

2.2 Class of Systems

Throughout this chapter, we are concerned with the analysis of nonlinear of the form

$$\Sigma \ : \ \begin{cases} \dot{x}(t) &= a(x(t)) + b(x(t))v(t), \\ z(t) &= h(x(t)) \end{cases} \tag{2.1}$$

where $x(0) = x \in \mathbf{R}^n$ is the initial state, $x(t) \in \mathbf{R}^n$ is the state at time t, $v(t) \in \mathbf{R}^p$ is the disturbance, and $z(t) \in \mathbf{R}^q$ is the output. To simplify notation, the unpenalized running cost will be denoted by $c(x) = |h(x)|^2$. The flow or trajectory of system Σ will be written as $\varphi(t, t_0, x_0; v)$, where x_0 is the initial state corresponding to the initial time t_0, t is the current time, and v is the disturbance. Where the initial state, initial time, and disturbance are clear from the context, this will be abbreviated to $x(t)$.

A number of definitions of reachability of the state space of systems are required.

Definition 2.2.1 (Complete Reachability) *A subset X of the state space of system Σ is completely reachable if for any x',$x'' \in X$, there exists a time horizon $0 \leq T < \infty$ and a disturbance $v \in \mathcal{L}_2[0,T]$ such that $\varphi(T,0,x';v) = x''$ with $\varphi(t,0,x';v) \in X$ for all $t \in [0,T]$.*

Definition 2.2.2 (Uniform Complete Reachability) *A subset X of the state space of system Σ is uniformly completely reachable if there exists a finite nonnegative mapping $T_{x,y} : X \times X \to \mathbf{R}$ and a mapping $v_{x,y}(s) : X \times X \times \mathbf{R} \to \mathbf{R}^p$ such that for any x, $y \in X$, $v_{x,y}(\cdot) \in \mathcal{L}_2[0,T_{x,y}]$, $\varphi(T_{x,y},0,x;v_{x,y}) = y$, and $T_{x,y}$, $\|v_{x,y}\|_{\mathcal{L}_2[0,T_{x,y}]}$ are bounded on compact subsets of $X \times X$.*

The following definition of local uniform reachability is a generalization of that found in [2].

Definition 2.2.3 (Local Uniform Reachability) *The state space of system Σ is defined to be locally uniformly reachable if for each $x' \in \mathbf{R}^n$, there exists a $\delta > 0$, and continuous functions $\alpha_1 : [0,\delta) \to \mathbf{R}^+$ and $\alpha_2 : [0,\delta) \to \mathbf{R}^+$ where $\alpha_1(0) = 0 = \alpha_2(0)$, such that for any $x'' \in \mathbf{R}^n$ with $\|x'' - x'\| < \delta$, there exists finite time $T \geq 0$ and a disturbance $v \in \mathcal{L}_2[0,T]$ such that $\varphi(T,0,x';v) = x''$ and*

$$\begin{aligned} \|v\|_{\mathcal{L}_2[0,T]} &\leq \alpha_1(|x'' - x'|), \\ T &\leq \alpha_2(|x'' - x'|). \end{aligned} \tag{2.2}$$

Where necessary, the class of systems Σ is further restricted by the following assumptions.

(A1) a, b, and c are continous functions.

(A2) The state space is Completely Reachable (Definition 2.2.1).

(A3) The state space is Uniformly Completely Reachable (Definition 2.2.2).

(A4) The state space is Locally Uniformly Reachable (Definition 2.2.3).

(A5) $|a(x) - a(y)| \leq L_1|x - y|$ for all x,$y \in \mathbf{R}^n$.

(A6) $|a(x)| \leq L_1(1 + |x|)$ for all $x \in \mathbf{R}^n$.

(A7) $a(x) \cdot x \leq -C_1|x|^2 + C_2$, for all $x \in \mathbf{R}^n$, where $C_1 > 0$, $C_2 \geq 0$.

(A8) $a(x) \cdot x \geq -C_3|x|^2 - C_4$, for all $x \in \mathbf{R}^n$, where $C_3 > 0$, $C_4 \geq 0$.

(A9) $|b(x) - b(y)| \leq L_2|x - y|$ for all $x, y \in \mathbf{R}^n$.

(A10) $|b(x)| \leq L_3$ for all $x \in \mathbf{R}^n$.

(A11) $|c(x) - c(y)| \leq L_4|x - y|$ for all $x, y \in \mathbf{R}^n$.

(A12) $0 \leq c(x) \leq L_5(1 + |x|^2)$ for all $x \in \mathbf{R}^n$.

(A13) $c(x) \geq L_6(|x|^2 - \mu)$ for all $x \in \mathbf{R}^n$, where $L_6 > 0$, $\mu \geq 0$.

(A14) $c(x) \to \infty$ as $|x| \to \infty$.

(A15) The level sets of $c(\cdot)$ are compact.

When required for the proof of particular results, these assumptions will be cited individually. Note that some of the above assumptions follow from others, but are listed individually for clarity.

The continuity and Lipschitz assumptions (A1) and (A5) are required for existence of solutions to the differential equation (2.1).

Proposition 2.2.4

(i) Assumptions (A1) and (A5) imply assumption (A6).

(ii) Assumptions (A1) and (A11) imply assumption (A12).

(iii) Assumption (A6) implies assumption (A8).

Proof: Applying assumption (A5),

$$\begin{aligned} |a(x)| &= |a(x) - a(y) + a(y)| \\ &\leq |a(x) - a(y)| + |a(y)| \\ &\leq L_1|x - y| + |a(y)| \\ &\leq \max(L_1, |a(y)|)(1 + |x - y|). \end{aligned}$$

Continuity assumption (A1) implies that for any $y \in \mathbf{R}^n$, $|a(y)| < \infty$. Choosing $y = 0$ in the above inequality yields assumption (A6). Hence, assumptions (A1) and (A5) imply (A6), which proves *(i)*. The same argument for $c(x)$ yields *(ii)*.

Also,

$$\begin{aligned} a(x)\cdot x &\geq -|a(x)||x| \\ &\geq -L_1(1+|x|)|x| \\ &\geq -\left(\frac{L_1}{2}\right)-\left(\frac{3L_1}{2}\right)|x|^2, \end{aligned}$$

using assumption (A6) and the fact that $|x|^2-2|x|+1\geq 0$. But, this is just assumption (A8). Hence, assumption (A6) implies (A8), which proves *(iii)*. ■

Assumption (A7) is a stability assumption that ensures the existence of an attracting set for the unperturbed dynamics of system (2.1).

Assumptions (A9) and (A10) are Lipschitz and boundedness assumptions respectively, and consequently independent.

Assumption (A13) provides a lower bound for the growth of the unpenalized running cost for the system. Assumption (A14) ensures that the state of the system cannot tend to ∞ while incurring only a finite unpenalized running cost, whilst assumption (A15) ensures that sequences of states with uniformly bounded unpenalized running cost are confined to a closed and bounded set. Assumptions (A14) and (A15) may be regarded as effectively detectability assumptions.

Proposition 2.2.5

(i) Assumption (A13) implies assumption (A14).

(ii) Assumption (A13) implies assumption (A15).

(iii) Assumptions (A1) and (A14) imply assumption (A15).

Proof: The proof of *(i)* is immediate.

To prove *(ii)*, define the level sets

$$\begin{aligned} K_L &= \{x\in\mathbf{R}^n : c(x)\leq L\}, \\ M_L &= \{x\in\mathbf{R}^n : L_6(|x|^2-\mu)\leq L\}. \end{aligned} \tag{2.3}$$

Suppose that assumption (A13) holds and $x\in K_L$. Then, $c(x)\leq L$. But, (A13) implies that $L_6(|x|^2-\mu)\leq c(x)\leq L$. That is, $x\in M_L$. Hence, $K_L\subseteq M_L$. But, M_L is compact by definition, so that K_L must be bounded. Furthermore, K_L is closed by definition. Hence, K_L is also compact. Consequently, assumption (A13) implies assumption (A15), which proves *(ii)*.

Finally, suppose that assumption (A14) holds. That is, $c(x) \to \infty$ as $|x| \to \infty$. So, given $M > 0$, there exists $X \geq 0$ such that

$$|x| > X \quad \Rightarrow \quad c(x) > M.$$

Contrapositively,

$$c(x) \leq M \quad \Rightarrow \quad |x| \leq X.$$

Hence, applying the definition of K_M (2.3), $x \in K_M$ implies that $|x| \leq X$. That is, $K_M \subseteq B[0;X]$. Now, choose any L such that $\inf_{x \in \mathbf{R}^n}\{c(x)\} \leq L \leq M$. $x \in K_L$ implies that $c(x) \leq L \leq M$. Hence, $K_L \subseteq K_M \subseteq B[0;X]$. That is, K_L is a bounded set. Suppose that assumption (A1) holds, so that $c(\cdot)$ is continuous. Then, K_L is closed by definition. Hence, K_L is compact. So, this demonstrates that K_L is compact for any $L \leq M$, where M is fixed arbitrarily large. That is, assumption (A15) holds.

So, assumptions (A1) and (A14) imply (A15), proving *(iii)*. ■

If the disturbance appears explicitly in each component of the state equation of system (2.1), intuitive it is expected that the state space of the system will be reachable. The following simple result states that indeed the three defined forms of reachability hold (Definitions 2.2.1, 2.2.2, and 2.2.3).

Theorem 2.2.6 *Suppose that system Σ satisfies assumption (A6) with state equation of the form*

$$\dot{x} = a(x) + bv,$$

where $b \in \mathbf{R}$ is a nonzero constant. Then, reachability assumptions (A2), (A3), and (A4) hold.

Proof: Let M be any compact subset of the state space of system Σ, $\mathbf{R}^n$. Let x, $y \in M$, and define the trajectory

$$x(s) = x + \left(\frac{y-x}{\sqrt{|y-x|}}\right) s. \tag{2.4}$$

Clearly $x(0) = x$ and $x(\sqrt{|y-x|}) = y$. So, define

$$T(x,y) = \sqrt{|y-x|}. \tag{2.5}$$

From (2.4),

$$\dot{x}(s) = \frac{y-x}{\sqrt{|y-x|}},$$

so that

$$v(x,y,s) \;=\; \frac{1}{b}\left[\frac{y-x}{\sqrt{|y-x|}} - a\left(x + \left(\frac{y-x}{\sqrt{|y-x|}}\right)s\right)\right]. \tag{2.6}$$

That is, $v(x,y,s)$ drives the system from state x to y in time $T(x,y)$. Now, applying the triangle inequality and assumption (A6),

$$\begin{aligned} |v(x,y,s)| &\leq \frac{1}{|b|}\left[\left|\frac{y-x}{\sqrt{|y-x|}}\right| + \left|a\left(x + \left(\frac{y-x}{\sqrt{|y-x|}}\right)s\right)\right|\right] \\ &\leq \frac{1}{|b|}\left[\sqrt{|y-x|} + L_1\left(1 + |x|^2 + \sqrt{|y-x|}s\right)\right] \\ &\leq \frac{1}{|b|}\left[\sqrt{|y-x|} + L_1\left(1 + |x|^2 + |x-y|\right)\right] \\ &=: L(x,y), \end{aligned}$$

for all $s \in [0, T(x,y)]$. Clearly $L(x,y)$ is uniformly bounded on compact sets. Hence,

$$\|v(x,y,\cdot)\|^2_{\mathcal{L}_2} \;\leq\; L(x,y)^2\sqrt{|y-x|} \;<\; \infty, \tag{2.7}$$

$$T(x,y) \;=\; \sqrt{|y-x|} \;<\; \infty, \tag{2.8}$$

for all x, $y \in M$. Inequalities (2.7) and (2.8) immediately imply assumption (A2). Since these inequalities hold for any x, y in any compact set $M \subset \mathbf{R}^n$, the state space of Σ must also be uniformly completely reachable, (A3). Furthermore, (2.7) and (2.8) define the continuous functions α_1 and α_2 in the inequality (2.2) of Definition 2.2.3. Hence, the state space of Σ is also locally uniformly reachable, (A4). ∎

2.3 The $\mathcal{L}_2$ and $\mathcal{FP}$ spaces

Conventional $\mathcal{H}_\infty$ theory is concerned with the analysis of systems with respect to an input / output energy ($\mathcal{L}_2$-) gain property. Consequently, the fundamental normed linear space is the $\mathcal{L}_2$ function space.

Definition 2.3.1 *The norm $\|\cdot\|_{\mathcal{L}_2}$ is defined to be the energy of a signal. That is,*

$$\|v\|_{\mathcal{L}_2} \;=\; \sqrt{\lim_{T\to\infty}\left\{\int_0^T |v(s)|^2\, ds\right\}}. \tag{2.9}$$

The space $\mathcal{L}_2$ is defined to be the space of all signals with finite $\mathcal{L}_2$ norm. That is,

$$\mathcal{L}_2 \;=\; \{v : \mathbf{R} \to \mathbf{R}^p \mid \|v\|_{\mathcal{L}_2} < \infty\}. \tag{2.10}$$

As power gain analysis of systems will be concerned with the analysis of systems with respect to an input / output power ($\mathcal{FP}$-) gain property, it is necessary to define a semi-normed linear $\mathcal{FP}$ (finite power) function space (see Proposition 2.3.5 for a proof of the seminorm property).

Definition 2.3.2 *The seminorm $\|\cdot\|_{\mathcal{FP}}$ is defined to be the average power of a signal. That is,*

$$\|v\|_{\mathcal{FP}} = \sqrt{\limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T |v(s)|^2\, ds\right\}}. \tag{2.11}$$

The space $\mathcal{FP}$ is defined to be the space of all signals with finite $\mathcal{FP}$ seminorm. That is,

$$\mathcal{FP} = \{v : \mathbf{R} \to \mathbf{R}^p \mid \|v\|_{\mathcal{FP}} < \infty\}. \tag{2.12}$$

Remark 2.3.3 With regard to the definitions of the $\mathcal{L}_2$ and $\mathcal{FP}$ spaces, it is important to note the use of a limit in (2.9) and a limit superior in (2.11). Since the integral $\int_0^T |v(s)|^2\, ds$ in (2.9) is a nondecreasing function of the horizon T, clearly the use of a limit in (2.9) is sufficient to ensure that the $\mathcal{L}_2$ norm is well defined. However, the fraction $\frac{1}{T}\int_0^T |v(s)|^2\, ds$ in (2.11) does not enjoy the same monotonicity property. Hence, a sequence $\{T_k\}$ which tends to infinity may result in multiple cluster points of this fraction, resulting in an undefined limit (see Example 2.3.4). To avoid this, a limsup is used in the definition of the seminorm (2.11), thereby capturing a tight upper bound for the power of the signal. ◀

Example 2.3.4 In this example, a signal $v \in \mathcal{FP}$ is constructed such that $P_v(T) := \frac{1}{T}\int_0^T |v(s)|^2\, ds$ evaluated along the unbounded increasing sequence $\{T_k\}$ has two cluster points.

Define $E_v(T) := \int_0^T \|v(s)\|^2\, ds$. Suppose that initially $P_v(T)$ is increasing towards an upper cluster point. For $P_v(T)$ to have a second cluster point, clearly $P_v(T)$ must decrease. Since $P_v(T) = \frac{E_v(T)}{T}$ and $E_v(T)$ is nondecreasing for all T, this requires that $E_v(T)$ increase at a rate slower than T. So, for the purpose of this example, choose v constant in some interval, as illustrated in Figure 2.1.

In order to have two cluster points of the sequence $P_v(T_k)$, the duration $T_3^k - T_2^k$ must be sufficiently large such that $P_v(T)$ attains a constant lower bound, for example $L_- = \frac{1}{2}$. This will be the lower cluster point. In view of Figure 2.1, the upper cluster

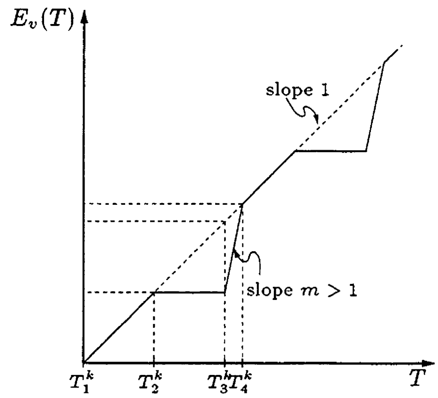


Figure 2.1: Behaviour of the integral $E_v(T) = \int_0^T |v(s)|^2\, ds$.

point must be be $L_+ = 1$. Choose $m = 2$. Then,

$$\begin{aligned} \frac{E_v(T_2^k)}{T_3^k} &= L_-, \\ E_v(T_2^k) &= T_2^k, \\ E_v(T_4^k) &= T_4^k, \\ m &= \frac{T_4^k - T_2^k}{T_4^k - T_3^k}. \end{aligned}$$

So, combining these equations yields

$$\begin{aligned} T_3^k &= 2T_2^k, \\ T_4^k &= 3T_2^k. \end{aligned}$$

The duration $T_2^k - T_1^k$ can assume any non-negative value. So choose

$$T_2^k - T_1^k = 1. \tag{2.13}$$

In summary, the times T_i^k are defined as

$$\begin{aligned} T_1^k &= T_4^{k-1}, \\ T_2^k &= T_1^k + 1, \\ T_3^k &= 2T_1^k + 2, \\ T_4^k &= 3T_1^k + 3. \end{aligned}$$

Assuming $T_1^k = 0$,

$$T_1^k = \frac{3(3^k - 1)}{2}. \tag{2.14}$$

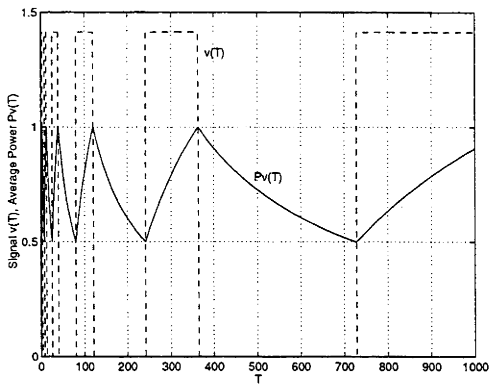


Figure 2.2: Finite horizon average power with two cluster points

As for the values of the function v during each time interval, since the slope of $E_v(T)$ is constant, v must be piecewise constant. Using the above equations for T_i^k, a signal v which yields two cluster points in the sequence $P_v(T_k)$ for any sequence $T_k \to \infty$ is given by

$$v(t) := \begin{cases} 0 & t < 0, \\ 1 & T_1^k \le t < T_1^k + 1, \\ 0 & T_1^k \le t < 2T_1^k + 2, \\ \sqrt{2} & 2T_1^k + 2 \le t < 3T_1^k + 3. \end{cases} \tag{2.15}$$

where $T_1^k = \frac{3(3^k-1)}{2}$ for all integers k. Figure 2.2 illustrates that $P_v(T)$ does indeed have two cluster points as $T \to \infty$. ◆

Due to the averaging properties of the operator $\|\cdot\|_{\mathcal{FP}}$, it is intuitively clear that $\|\cdot\|_{\mathcal{FP}}$ cannot define a norm (since $\|v\|_{\mathcal{FP}} = 0 \not\Rightarrow v \equiv 0$, see Proposition 2.3.7).

Proposition 2.3.5 **($\mathcal{FP}$-seminorm)** *The operator $\|\cdot\|_{\mathcal{FP}}$ defines a seminorm on $\mathcal{FP}$. That is, $(\mathcal{FP}, \|\cdot\|_{\mathcal{FP}})$ is a seminormed linear space.*

Proof: Check the three axioms for a seminorm.

(i) Let $w \in \mathcal{FP}$ and $\alpha \in \mathbf{R}$ be any scalar. Then,

$$\begin{aligned} \|\alpha\, w\|_{\mathcal{FP}}^2 &= \limsup_{T\to\infty} \left\{ \frac{1}{T} \int_0^T |\alpha\, w(t)|^2\, dt \right\} \\ &= \limsup_{T\to\infty} \left\{ \frac{1}{T} \int_0^T |\alpha|^2 |w(t)|^2\, dt \right\} \\ &= |\alpha|^2 \limsup_{T\to\infty} \left\{ \frac{1}{T} \int_0^T |w(t)|^2\, dt \right\} \\ &= |\alpha|^2 \|w\|_{\mathcal{FP}}^2. \end{aligned}$$

(ii) Let $v, w \in \mathcal{FP}$. Then,

$$\begin{aligned}
\|v+w\|_{\mathcal{FP}}^2 &= \limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T |v(t)+w(t)|^2\,dt\right\}\\
&\leq \limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T |v(t)|^2+2|v(t)||w(t)|+|w(t)|^2\,dt\right\}\\
&\leq \limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T |v(t)|^2\,dt\right\}+2\limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T |v(t)||w(t)|\,dt\right\}+\\
&\qquad \limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T |w(t)|^2\,dt\right\}\\
&= \|v\|_{\mathcal{FP}}^2+2\limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T |v(t)||w(t)|\,dt\right\}+\|w\|_{\mathcal{FP}}^2. \qquad (2.16)
\end{aligned}$$

Now, Hölder's Inequality states that

$$\left(\int_I f\;g\right) \leq \left(\int_I f^p\right)^{\frac{1}{p}}\left(\int_I g^q\right)^{\frac{1}{q}}, \qquad (2.17)$$

provided $\frac{1}{p}+\frac{1}{q}=1$. Choosing $p=q=2$,

$$\frac{1}{T}\int_0^T |v(t)||w(t)|\,dt \;\leq\; \left(\frac{1}{T}\int_0^T |v(t)|^2\,dt\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T |w(t)|^2\,dt\right)^{\frac{1}{2}}. \qquad (2.18)$$

Also, $\limsup(a\;b) \leq \limsup(a)\limsup(b)$. Hence combining inequalities (2.16) and (2.18) yields that

$$\begin{aligned}
\|v+w\|_{\mathcal{FP}}^2 &\leq \|v\|_{\mathcal{FP}}^2+2\|v\|_{\mathcal{FP}}\|w\|_{\mathcal{FP}}+\|w\|_{\mathcal{FP}}^2\\
&= \left(\|v\|_{\mathcal{FP}}+\|w\|_{\mathcal{FP}}\right)^2.
\end{aligned}$$

That is,

$$\|v+w\|_{\mathcal{FP}} \;\leq\; \|v\|_{\mathcal{FP}}+\|w\|_{\mathcal{FP}}.$$

(iii) $\frac{1}{T}\int_0^T \|w(t)\|^2\,dt \geq 0$ for all $w \in \mathcal{FP}$, $T>0$. So,

$$\|w\|_{\mathcal{FP}} \;\geq\; 0$$

for all $w \in \mathcal{FP}$.

Hence, since all three axioms are satisfied, $\|\cdot\|_{\mathcal{FP}}$ must be a seminorm. ∎

Note that the remaining norm property which fails is the existence of a unique zero power signal. The following definition and proposition deomnstrate this lack of uniqueness by showing that the $\mathcal{L}_2$ space is in the kernel of the $\mathcal{FP}$-seminorm.

Definition 2.3.6 ($\mathcal{FP}_0$) *The space of all zero power signals is denoted by*

$$\mathcal{FP}_0 \;:=\; \{v \in \mathcal{FP} : \|v\|_{\mathcal{FP}}=0\}. \qquad (2.19)$$

Proposition 2.3.7

(i) $\mathcal{L}_2 \subset \mathcal{FP}$ *and* $v \in \mathcal{L}_2 \Rightarrow \|v\|_{\mathcal{FP}} = 0$,

(ii) $\mathcal{L}_2 \subset \mathcal{FP}_0$,

(iii) $\mathcal{FP} \subset \mathcal{L}_{2e}$.

Proof:

(i) Let $v \in \mathcal{L}_2$. Then, by (2.9),

$$0 \;\leq\; \lim_{T\to\infty}\left\{\int_0^T |v(t)|^2\,dt\right\} \;=\; K \;<\; \infty.$$

Given $\varepsilon > 0$, there exists a $T^* > 0$ such that

$$\begin{aligned} T > T^* \quad &\Rightarrow \quad 0 \leq \int_0^T |v(t)|^2\,dt - K < \varepsilon \\ &\Leftrightarrow \quad 0 \leq K \leq \int_0^T |v(t)|^2\,dt < K + \varepsilon. \end{aligned}$$

Dividing by T,

$$0 \leq \frac{K}{T} \leq \frac{1}{T}\int_0^T |v(t)|^2\,dt < \frac{K+\varepsilon}{T}.$$

As $T \to \infty$, $\frac{K}{T} \to 0$ and $\frac{K+\varepsilon}{T} \to 0$. Hence,

$$\limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T |v(t)|^2\,dt\right\} \;=\; 0.$$

That is, $\|v\|_{\mathcal{FP}} = 0$. It is easy to show that $\mathcal{FP} \setminus \mathcal{L}_2 \neq \emptyset$ by showing that any constant signal has finite power, but infinite energy.

(ii) From *(i)* and Definition 2.3.6, $v \in \mathcal{L}_2$ implies that $v \in \mathcal{FP}_0$. Hence, $\mathcal{L}_2 \subseteq \mathcal{FP}_0$. To show that $\mathcal{FP}_0 \setminus \mathcal{L}_2 \neq \emptyset$, consider the signal

$$v(t) \;:=\; \begin{cases} \frac{1}{(1+t)^{\frac{1}{4}}} & t \geq 0, \\ 0 & t < 0. \end{cases}$$

Then,

$$\int_0^T \|v(t)\|^2\,dt \;=\; 2\left(\sqrt{T+1} - 1\right).$$

So, clearly $\|v\|_{\mathcal{L}_2} = \infty$, while $\|v\|_{\mathcal{FP}}$ is zero.

(iii) Suppose that $\mathcal{FP} \not\subseteq \mathcal{L}_{2e}$. Then, there exists a $v \in \mathcal{FP}$ and an $r \geq 0$ such that

$$\int_0^r |v(s)|^2\,ds \;=\; \infty.$$

But, from (2.11),

$$\|v\|_{\mathcal{FP}}^2 \geq \limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T |v(s)|^2\, ds\right\} = \infty,$$

which implies that $v \notin \mathcal{FP}$, and is thus a contradiction. Hence, $\mathcal{FP} \subseteq \mathcal{L}_{2e}$. Next, consider the signal $v(t) = t$. For any finite T, $\int_0^T |v(s)|^2\, ds < \infty$, which implies that $v \in \mathcal{L}_{2e}$. But, $\|v\|_{\mathcal{FP}} = \infty$, which implies that $v \in \mathcal{L}_{2e} \setminus \mathcal{FP}$. That is, $\mathcal{FP} \subset \mathcal{L}_{2e}$.

■

2.4 The Power Gain Inequality

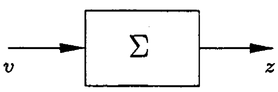


Figure 2.3: Input/Output View of Σ

Before defining the notion of power gain, recall the definition of the $\mathcal{L}_2$-gain of a system.

Definition 2.4.1 *System Σ has finite energy ($\mathcal{L}_2$-) gain $\leq \gamma$ if there exists a finite nonnegative function $\beta(x)$ such that*

$$\int_0^T |z(s)|^2\, ds \leq \gamma^2 \int_0^T |v(s)|^2\, ds + \beta(x) \tag{2.20}$$

holds for all $v \in \mathcal{L}_2[0,T]$, all $T \geq 0$, and all $x \in \mathbf{R}^n$, where x is the initial state of Σ. The $\mathcal{L}_2$-gain γ^* of a system is also the $\mathcal{H}_\infty$ norm. That is, $\gamma^* = \|\Sigma\|_{\mathcal{H}_\infty}$.

By inspection of (2.20), the $\mathcal{L}_2$-gain inequality is an input / output energy relationship. With $v \equiv 0$, the energy of z is bounded for all $T \geq 0$. Hence, systems with infinite energy output (such as limit cycle systems) cannot have $\mathcal{L}_2$-gain. However, systems with infinite energy, finite power outputs can be treated using a generalization of (2.20).

The power gain of system Σ is defined as an input/output property which relates the output power to the input power, with the system starting from a given initial state. This is formalized in the following definition.

Definition 2.4.2 *System Σ has finite power ($\mathcal{FP}$-) gain $\leq \gamma$ if there exists a finite nonnegative pair $(\lambda(x), \beta(x))$ such that*

$$\int_0^T |z(s)|^2 \, ds \;\; \leq \;\; \gamma^2 \int_0^T |v(s)|^2 \, ds + \lambda(x)T + \beta(x) \tag{2.21}$$

holds for all $v \in \mathcal{L}_2[0,T]$, all $T \geq 0$, and all $x \in \mathbf{R}^n$, where x is the initial state of Σ.

In the above definition, λ is referred to as the *power bias* and β as the *energy bias.* Inspection of (2.21) reveals the similarity between $\mathcal{FP}$-gain and the standard notion of $\mathcal{L}_2$-gain (2.20). This similarity is expressed in the following proposition.

Proposition 2.4.3 *Any system with $\mathcal{FP}$-gain $\leq \gamma$ and zero power bias has $\mathcal{L}_2$-gain $\leq \gamma$. Conversely, any system with $\mathcal{L}_2$-gain $\leq \gamma$ has $\mathcal{FP}$-gain $\leq \gamma$ with zero power bias.*

Proof: Suppose Σ has $\mathcal{FP}$-gain $\leq \gamma$ with zero power bias. Applying Definition 2.4.2, there exists a finite nonnegative function $\beta(x)$ (corresponding to $\lambda = 0$) such that (2.21) holds,

$$\int_0^T |z(s)|^2 \, ds \;\; \leq \;\; \gamma^2 \int_0^T |v(s)|^2 \, ds + \beta(x)$$

for all $v \in \mathcal{L}_2[0,T]$, $T \geq 0$. But, this is just (2.20). That is, Σ has $\mathcal{L}_2$-gain $\leq \gamma$.

For the converse, suppose that Σ has $\mathcal{L}_2$-gain $\leq \gamma$. Then (2.20) holds for some finite nonnegative function $\beta(x)$. But, this is just (2.21) with $\lambda = 0$. Hence, system Σ has $\mathcal{FP}$-gain $\leq \gamma$ with power bias / energy bias pair (λ, β), where $\lambda = 0$. ■

For systems with $\mathcal{FP}$-gain, the direction of the inequality in (2.21) implies that the choice of power bias is nonunique. That is, it is apparent that substitution of power bias λ with another λ' where $\lambda' > \lambda$ does not alter the statement of $\mathcal{FP}$-gain. However, the nature of the inequality (2.21) does suggest the existence of a minimal power bias. This issue will be dealt with when we define the concept of *available power* in Section 2.6.

In addition to the power bias λ, inequality (2.21) defines a gain factor γ. A system Σ with $\mathcal{FP}$-gain $\leq \gamma$ with power bias λ must have the property that γ is bounded below by the *power gain* γ_λ^* of the system, where

$$\gamma_\lambda^* \;\; = \;\; \inf\{\gamma \geq 0 : (2.21) \text{ holds}\}. \tag{2.22}$$

Note that the dependence of γ_λ^* on the power bias λ is an immediate departure from the conventional notion of gain in $\mathcal{H}_\infty$ theory. For systems with $\mathcal{L}_2$-gain, clearly γ_λ^*

reduces to the $\mathcal{H}_\infty$ norm. However, in the case of power gain with nonzero power bias, we expect intuitively that a larger choice of power bias λ may lead to a reduced power gain γ_λ^*. This is due to the transfer of energy from the $\int_0^T |v(s)|^2\, ds$ term to the λT term in inequality (2.21).

Example 2.4.4 In Section 5.4, the scalar linear system with output saturation (5.53) is shown to have power gain

$$\gamma_\lambda^* = \begin{cases} \left|\frac{b}{a\varepsilon}\right| \sqrt{c^2\varepsilon^2 - \lambda} & \lambda \in [0, c^2\varepsilon^2), \\ 0 & \lambda \in [c^2\varepsilon^2, \infty), \end{cases} \tag{2.23}$$

where a, b, c, ε are parameters defining the drift and output of the system. Note that $\varepsilon = \infty$ corresponds to a linear system. As expected, (2.23) reduces to $\left|\frac{bc}{a}\right|$ with $\varepsilon = \infty$, which is precisely the $\mathcal{H}_\infty$-norm of the resulting linear system. ♦

Setting $T = 0$ in (2.21), it is clear that the energy bias $\beta(x)$ is the output energy attributable to transients arising from the initial state x. Similarly, the power bias may be interpreted as a measure of the power attributable to transients arising from the initial state x *and* the dynamics arising from the application of any energy signal. This becomes apparent when both sides of (2.21) is divided by $T > 0$, with $T \to \infty$. Then, applying the definition (2.11) of the $\mathcal{FP}$-seminorm,

$$\|z\|_{\mathcal{FP}}^2 \leq \gamma^2 \|v\|_{\mathcal{FP}}^2 + \lambda(x). \tag{2.24}$$

Applying Proposition 2.3.7, any energy signal must have zero $\mathcal{FP}$-seminorm. Hence, in the absence of power disturbances,

$$\|z\|_{\mathcal{FP}}^2 \leq \lambda(x). \tag{2.25}$$

That is, the power bias is a bound for the output power under the application of any energy disturbances.

Remark 2.4.5 Immediately it is evident that for detectable systems with nonzero power bias, the state need not be asymptotically stable in the absence of disturbances. For example, the state of the system may be oscillatory or even chaotic. That is, systems with nonzero power bias can exhibit *internal power generation.* Examples of such systems are considered in Chapter 5 (Sections 5.7 and 5.8 for example). ◄

In defining the notion of $\mathcal{FP}$-gain, clearly inequality (2.24) is an alternative to the stated definition (2.21). Although (2.24) is less restrictive than (2.21) with regard to system behaviour for small t, (2.24) proves difficult to later link to the notion of *dissipativity*. The difficulty arises from the fact that although (2.21) implies (2.24), the converse is not necessarily true. Consequently, inequality (2.21) is retained as the definition for power gain.

Example 2.4.6 The disturbance free cubic drift system of Section 5.5 satisfies the inequality (2.24) with $\lambda = 0$. However, no finite, lower bounded function $\beta(x)$ exists such that (2.21) holds for all $v \in \mathcal{L}_2[0,T]$ and all $T \geq 0$. ♦

Finally, it is important to understand the relationship between the concepts of $\mathcal{FP}$-gain and stability. As already mentioned, systems with $\mathcal{FP}$-gain need not be asymptotically stable. So, in the remainder of this section, we demonstrate that under some detectability assumptions, a system with the $\mathcal{FP}$-gain property enjoys stability in the sense that unperturbed trajectories tend to a compact set. Conversely, we demonstrate that a large class of systems which exhibit such a stability property automatically exhibit the $\mathcal{FP}$-gain property.

Theorem 2.4.7 *Suppose that the growth assumption (A6) and detectability assumptions (A14) and (A15) hold, and that the system Σ has $\mathcal{FP}$-gain with strictly positive power bias λ, independent of the initial state. Then, Σ is stable in the sense that unperturbed trajectories tend to a compact set.*

Proof: In the absence of disturbances, the power gain inequality (2.21) implies that

$$\int_0^T |c(x(s))|^2 \, ds \;\leq\; \lambda T + \beta(x),$$

where $\lambda > 0$.

Given power bias $\lambda > 0$, define the level set $K_{3\lambda} = \{x \in \mathbf{R}^n : c(x) \leq 3\lambda\}$. By assumption (A15), $K_{3\lambda}$ is compact. Now supppose that the trajectory $x(s)$ never enters the set $K_{3\lambda}$. Then,

$$\begin{aligned} \int_0^T c(x(s)) \, ds \;&>\; 3\lambda T \\ &>\; 2\lambda T + \beta(x) \end{aligned} \tag{2.26}$$

for all $T > \frac{\beta(x)}{\lambda}$. But, system Σ has $\mathcal{FP}$-gain $\leq \gamma$ with power bias λ. Hence, in the absence of disturbances, power gain inequality (2.21) implies that

$$\int_0^T |c(x(s))|^2 \, ds \;\leq\; \lambda T + \beta(x). \tag{2.27}$$

But, (2.26) and (2.27) together yield a contradiction. Hence, $x(s)$ enters the set $K_{3\lambda}$ sometime in the interval $[0, \frac{\beta(x)}{\lambda}]$. Once $x(s)$ has entered $K_{3\lambda}$, it may exit again. However, applying the same argument, $x(s)$ must reenter the $K_{3\lambda}$. If the exit time is τ, then reetry must occur sometime in the interval $[\tau, \tau + T_e]$, where T_e is the maximum time for the trajectory to enter the set $K_{3\lambda} \setminus \partial K_{3\lambda}$ when the initial state $x = x(0)$ is anywhere on the boundary of $K_{3\lambda}$. That is, $T_e \leq \max_{x \in \partial K_{3\lambda}} \left\{ \frac{\beta(x)}{\lambda} \right\} < \infty$ since $K_{3\lambda}$ is compact.

Summarizing thus far, the trajectory enters a compact set in finite time. Furthermore, if the trajectory leaves the compact set, then it must reenter in finite time. It remains to be shown that the excursions from the compact set are bounded.

Since the system is unperturbed,

$$x(t) \;=\; x + \int_0^t a(x(s))\, ds.$$

So, taking the norm of both sides and applying assumption (A6),

$$\begin{aligned} |x(t)| &\leq |x| + \int_0^t L_1(1 + |x(s)|)\, ds \\ &= (|x| + L_1 t) + L_1 \int_0^t |x(s)|\, ds. \end{aligned}$$

Applying Gronwall's Inequality implies that

$$|x(t)| \;\leq\; (|x| + L_1 t)\, \mathrm{e}^{L_1 t}$$

Choosing $x \in \partial K_{3\lambda}$, we know that $t \in [0, T_e]$. Hence,

$$\begin{aligned} |x(t)| &\leq \left(\max_{x \in \partial K_{3\lambda}} \{|x|\} + L_1 T_e \right) \mathrm{e}^{L_1 T_e} \\ &=: R < \infty. \end{aligned}$$

Hence, all excursions from $K_{3\lambda}$ are contained within the compact set $B_R := \{x \in \mathbf{R}^n : |x| \leq R\}$. Furthermore, since $K_{3\lambda} \subseteq B_R$, all trajectories of the unperturbed system Σ tend to the compact set B_R independently of the initial state $x \in \mathbf{R}^n$. ■

Theorem 2.4.8 *Suppose that assumptions (A7), (A10), and (A12) hold. Then, system Σ has finite power gain with*

$$\int_0^T |z(s)|^2\, ds \;\leq\; \frac{2L_3L_5}{C_1\delta} \int_0^T |v(s)|^2\, ds + L_5\left(1 + \frac{2C_2}{C_1}\right) T + \frac{L_5}{C_1}|x|^2 \tag{2.28}$$

for all $v \in \mathcal{L}_2[0,T]$ and all $x \in \mathbf{R}^n$, where δ is sufficiently small such that $L_3\delta \leq 2C_1$.

Proof: As in [15], we begin by letting $Q(x) = \frac{1}{2}|x|^2$. Differentiating Q along the trajectory of the system, $x(t)$, and applying assumptions (A7) and (A10),

$$\begin{aligned}\frac{d}{dt}Q(x(t)) &= \dot{x}(t)\cdot x(t)\\ &= a(x(t))\cdot x(t) + (x(t))^T b(x(t))v(t)\\ &\leq a(x(t))\cdot x(t) + L_3|x(t)||v(t)|\\ &\leq -C_1|x(t)|^2 + C_2 + L_3|x(t)||v(t)|\end{aligned}$$

For any $\delta > 0$,

$$|x(t)||v(t)| \leq \frac{\delta}{4}|x(t)|^2 + \frac{1}{\delta}|v(t)|^2.$$

Hence, by choosing $\delta > 0$ such that $L_3\delta \leq 2C_1$,

$$\begin{aligned}\frac{d}{dt}Q(x(t)) &\leq -\frac{C_1}{2}|x(t)|^2 + C_2 + \frac{L_3}{\delta}|v(t)|^2\\ &= -C_1 Q(x(t)) + C_2 + \frac{L_3}{\delta}|v(t)|^2.\end{aligned}$$

Integrating with respect to t,

$$Q(x(t)) \leq Q(x)e^{-C_1 t} + \int_0^t e^{-C_1(t-s)}\left(C_2 + \frac{L_3}{\delta}|v(s)|^2\right) ds \tag{2.29}$$

for all $t \geq 0$. Hence,

$$\int_0^T |x(t)|^2\, dt \leq \int_0^T |x|^2 e^{-C_1 t}\, dt + 2\int_0^T\int_0^t e^{-C_1(t-s)}\left(C_2 + \frac{L_3}{\delta}|v(s)|^2\right) ds\, dt \tag{2.30}$$

for all $T \geq 0$. The second term on the RHS of (2.30) can be simplifed by swapping the order of integration:

$$\begin{aligned}&\int_0^T\int_0^t e^{-C_1(t-s)}\left(C_2 + \frac{L_3}{\delta}|v(s)|^2\right) ds\, dt\\ &= \int_0^T\int_s^T e^{-C_1(t-s)}\left(C_2 + \frac{L_3}{\delta}|v(s)|^2\right) dt\, ds\\ &= \int_0^T e^{C_1 s}\left(C_2 + \frac{L_3}{\delta}|v(s)|^2\right)\int_s^T e^{-C_1 t}\, dt\, ds\\ &= \int_0^T \left(\frac{C_2}{C_1} + \frac{L_3}{C_1\delta}|v(s)|^2\right)[1 - e^{-C_1(T-s)}]\, ds\\ &\leq \frac{C_2}{C_1}T + \frac{L_3}{C_1\delta}\int_0^T |v(s)|^2\, ds.\end{aligned}$$

So, (2.30) now becomes

$$\int_0^T |x(s)|^2\, ds \leq \frac{2L_3}{C_1\delta}\int_0^T |v(s)|^2\, ds + \frac{2C_2}{C_1}T + \frac{1}{C_1}|x|^2. \tag{2.31}$$

Applying assumption (A12), the output energy and the state energy are related by the inequality

$$\int_0^T |z(s)|^2 \, ds \;\leq\; L_5 \int_0^T |x(s)|^2 \, ds + L_5 T. \tag{2.32}$$

Combining (2.31) and (2.32), yields the inequality (2.28). ∎

Theorem 2.4.8 provides sufficient conditions for a system to exhibit power gain. The power bias, energy bias, and gain parameters provided by this theorem are explicit in that they depend directly on the constants provided by the underlying assumptions. That is, Theorem 2.4.8 allows us to conclude that a system has power gain less than or equal to a particular fixed gain. As such, Theorem 2.4.8 is useful as a top level power gain analysis tool.

2.5 Finite Time Maximum Energy Retrieval from Nonlinear Systems

In order to develop more powerful power gain analysis tools for nonlinear systems, it is useful to reformulate the power gain inequality as a variational problem. That is, we can rewrite the power gain inequality as a bound on a *value function*. Then, the problem of determining if a system has power gain can be solved by computing this value function.

2.5.1 Definition and Some General Properties

As with standard nonlinear $\mathcal{H}_\infty$ theory, it is possible to define a finite horizon value function which is a measure of the most energy that can be extracted from a system in a finite time, starting from a given initial state.

Definition 2.5.1 *Define the finite horizon value function*

$$V(x,T) \;=\; \sup_{v \in \mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2 |v(s)|^2 \right] \, ds : x(0) = x \right\}. \tag{2.33}$$

Given that the class of systems (Section 2.2) of interest include systems which exhibit the power gain property, it is feasible that this value function may be unbounded as the time horizon tends to infinity. That is, the total energy retrievable from the system may be infinite. This follows directly from the notion that systems with the power gain property can exhibit internal power generation (Remark 2.4.5).

In the standard $\mathcal{H}_\infty$ theory, the attendant $\mathcal{L}_2$-gain property implies uniform boundedness (with respect to the time horizon T) of this finite horizon value function $V(x,T)$ (2.33). Furthermore, due to the nondecreasing nature of $V(x,T)$ in T (Proposition 2.5.2), this implies that in the infinite horizon limit, $V(x,T)$ (2.33) converges to some stationary value function $V(x)$. This limit function $V(x)$ has special significance in dissipative systems theory [42, 21, 22] as the available storage for the system. So, immediately it is clear that the finite horizon value function $V(x,T)$ (2.33) is a fundamental definition in $\mathcal{H}_\infty$ theory. As will become clear, the finite horizon value function $V(x,T)$ (2.33) is also naturally a fundamental definition in the theory of power gain analysis of nonlinear systems.

To begin with, it is useful to establish some of the basic properties of $V(x,T)$ (2.33). Since $V(x,T)$ is representative of the most energy retrieved from a system over the time interval $[0,T]$, we expect that in increasing the time horizon, the energy retrieved cannot decrease.

Proposition 2.5.2 *The value function $V(x,T)$ is nondecreasing in T.*

Proof: Fix $\tau \leq T$, and define a switching disturbance

$$\tilde{v}(s) = \begin{cases} v(s) & s \leq \tau \\ 0 & s > \tau \end{cases},$$

with $v \in \mathcal{L}_2[0,\tau]$. Now, $\tilde{v}$ must be suboptimal in the definition of $V(x,T)$, so that

$$\begin{aligned} V(x,T) &\geq \int_0^T \left[c(\tilde{x}(s)) - \gamma^2|\tilde{v}(s)|^2\right] ds \\ &= \int_0^\tau \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds + \int_\tau^T c(\hat{x}(s))\, ds \\ &\geq \int_0^\tau \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds \end{aligned}$$

for all $v \in \mathcal{L}_2[0,\tau]$, where $\hat{x}(\cdot)$ is the disturbance free trajectory. Hence, taking the sup over v,

$$V(x,T) \geq V(x,\tau)$$

for any $T \geq \tau$. ■

The inclusion of a gain factor γ in the definition of $V(x,T)$ (2.33) facilitates discounting of energy in the output attributable to energy supplied to the system via the disturbance. The gain can then be naturally interpretated as a measure of how the system amplifies the disturbance energy as it flows through to the output. So, a reduction in the gain factor γ may be interpreted as attributing less energy to the disturbance, whilst attributing more to the stored energy in the system. Hence, a reduction in gain should imply an increase in the retrievable energy.

Proposition 2.5.3 *The value function $V_\gamma(x,T)$ is nonincreasing in γ, where $V_\gamma(x,T)$ is the finite horizon value function $V(x,T)$ (2.33) for gain γ.*

Proof: Let $\bar{\gamma} < \gamma$. Then,

$$\int_0^T \left[c(x(s)) - \bar{\gamma}^2|v(s)|^2\right] \, ds \;\; \geq \;\; \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right] \, ds$$

for all $v \in \mathcal{L}_2[0,T]$. So, taking the sup over v and applying Definition 2.5.1 yields that

$$V_{\bar{\gamma}}(x,T) \;\; \geq \;\; V_\gamma(x,T)$$

for all $x \in \mathbf{R}^n$ and all $T \geq 0$. ∎

These properties are fundamental in linking the power gain property with $V(x,T)$ (2.33).

2.5.2 V(x, T) and the Power Gain Inequality

By inspection of the power gain inequality (2.21) and the definition of the finite horizon value function $V(x,T)$ (2.33), it is clear that systems which possess the power gain property must have $V(x,T)$ bounded above by an affine function of the time horizon T. Note that in standard $\mathcal{H}_\infty$ theory, $V(x,T)$ is bounded above uniformly with respect to T (due to the $\mathcal{L}_2$-gain property).

Theorem 2.5.4 *System Σ has $\mathcal{FP}$-gain $\leq \gamma$ with power bias / energy bias pair $(\lambda(x), \beta(x))$ iff*

$$V_\gamma(x,T) \;\; \leq \;\; \lambda(x)T + \beta(x) \tag{2.34}$$

for all $T \geq 0$ and all $x \in \mathbf{R}^n$, where $V_\gamma(x,T)$ is the finite horizon value function $V(x,T)$ (2.33) for gain γ, and the pair $(\lambda(x), \beta(x))$ is finite and bounded below.

Proof: Suppose that system Σ has $\mathcal{FP}$-gain $\leq \gamma$ with power bias / energy bias pair $(\lambda(x), \beta(x))$. Then, rearranging the power gain inequality (2.21),

$$\int_0^T \left[|z(s)|^2 - \gamma^2|v(s)|^2\right] \, ds \;\; \leq \;\; \lambda(x)T + \beta(x)$$

for all $v \in \mathcal{L}_2[0,T]$, all $T \geq 0$, and all $x \in \mathbf{R}^n$. Taking the sup over $v \in \mathcal{L}_2[0,T]$ yields (2.34).

Conversely, suppose that (2.34) holds for all $T \geq 0$ and all $x \in \mathbf{R}^n$, and that the pair $(\lambda(x), \beta(x))$ is finite and bounded below. Then, choosing any $x \in \mathbf{R}^n$, any $T \geq 0$, and any suboptimal $v \in \mathcal{L}_2[0,T]$,

$$\int_0^T \left[c(x(s)) - \gamma^2 |v(s)|^2\right] \, ds \;\leq\; V_\gamma(x,T) \;\leq\; \lambda(x)T + \beta(x),$$

which is just the power gain inequality (2.21). ■

By assuming that there exists an attracting set (rather than a point) in the dynamics of system Σ, it is possible to prove (via the power gain property and simple properties of Section 2.5.1) that the finite horizon value function $V(x,T)$ (2.33) is bounded above by an affine function of T. Fundamental to this is the following refinement of Theorem 2.4.8.

Lemma 2.5.5 *Suppose that assumptions (A7) and (A10) hold. Then, there exists constants B_1, B_2 such that*

$$|x(t)|^2 \;\leq\; |x|^2 + B_1 t + B_2 \int_0^t |v(s)|^2 \, ds \tag{2.35}$$

for all $v \in \mathcal{L}_2[0,t]$, all $t \geq 0$, and all $x \in \mathbf{R}^n$. Furthermore, if assumption (A12) also holds, then there exists a gain $\bar{\gamma} \geq 0$ and constants B_3, $B_4 \geq 0$ such that

$$\int_0^T |z(s)|^2 \, ds \;\leq\; \bar{\gamma}^2 \int_0^T |v(s)|^2 \, ds + B_3 T + B_4 |x|^2 \tag{2.36}$$

for all $v \in \mathcal{L}_2[0,T]$, all $T \geq 0$, and all $x \in \mathbf{R}^n$.

Proof: Following the proof of Theorem 2.4.8, (2.29) states that

$$|x(t)|^2 \;\leq\; |x|^2 e^{-C_1 t} + \int_0^t e^{-C_1(t-s)} \left[C_2 + \frac{L_3}{\delta}|v(s)|^2\right] \, ds.$$

Since $t \geq 0$ and $C_1 > 0$, clearly $0 < e^{-C_1 t} \leq 1$. Furthermore, in the integral term, $0 < e^{-C_1(t-s)} \leq 1$ for all $s \in [0,t]$. Hence,

$$\begin{aligned} |x(t)|^2 &\leq |x|^2 + \int_0^t \left[C_2 + \frac{L_3}{\delta}|v(s)|^2\right] \, ds \\ &= |x|^2 + C_2 t + \frac{L_3}{\delta} \int_0^t |v(s)|^2 \, ds, \end{aligned}$$

which is precisely (2.35).

The second assertion is just inequality (2.28) from Theorem 2.4.8. ■

The power gain property resulting from Lemma 2.5.5 may then be applied to find the affine growth bound with respect to the time horizon T for the finite horizon value function $V(x,T)$ (2.33).

Theorem 2.5.6 *Suppose that assumptions (A7), (A10), and (A12) hold. Then, there exists a gain* $\bar{\gamma} \geq 0$ *such that for all gains* $\gamma \geq \bar{\gamma}$,

$$V_\gamma(x,T) \leq B_3 T + B_4 |x|^2 \tag{2.37}$$

Proof: By Lemma 2.5.5, there exists gain $\bar{\gamma} \geq 0$ and constants B_3, $B_4 \geq 0$ such that system Σ has $\mathcal{FP}$-gain $\leq \bar{\gamma}$ with power bias / energy bias pair $(B_3, B_4|x|^2)$. Hence, by Theorem 2.5.4,

$$V_{\bar{\gamma}}(x,T) \leq B_3 T + B_4 |x|^2, \tag{2.38}$$

where $V_{\bar{\gamma}}(x,T)$ is the finite horizon value function $V(x,T)$ (2.33) for gain $\bar{\gamma}$. Furthermore, by Proposition 2.5.3, $V_\gamma(x,T) \leq V_{\bar{\gamma}}(x,T)$ for all $\gamma \geq \bar{\gamma}$. Thus, (2.38) implies that

$$V_\gamma(x,T) \leq V_{\bar{\gamma}}(x,T) \leq B_3 T + B_4 |x|^2$$

for all $\gamma \geq \bar{\gamma}$. ∎

2.5.3 Dynamic Programming

In the preceding section, the problem of determining power gain was reformulated (Theorem 2.5.4) in terms of a variational problem defined by the finite horizon value function $V(x,T)$ (2.33). Consequently, the test for power gain now involves the solution of a variational problem. Such variational problems are readily solved via dynamic programming [28, 17].

Before proving a dynamic programming principle for the finite horizon value function $V(x,T)$ (2.33), the following definition of near optimal disturbance is required.

Definition 2.5.7 *Given* $\delta > 0$, *the disturbance* $v^\delta \in \mathcal{L}_2[0,T]$ *is* δ*-optimal in the definition of the finite horizon value function* $V_\gamma(x,T)$ *(2.33) for initial state* x, *horizon* T, *and gain* γ *if*

$$V(x,T) - \delta \leq \int_0^T \left[c(x^\delta(s)) - \gamma^2 |v(s)|^2 \right] ds. \tag{2.39}$$

Using this definition of near optimal disturbances, the dynamic programming principle for $V(x,T)$ (2.33) can be established [17] without assuming existence of the optimal disturbance.

Theorem 2.5.8 *Given horizon $T \geq 0$ and initial state $x \in \mathbf{R}^n$, the finite horizon value function $V(x,T)$ (2.33) satisfies the dynamic programming equation*

$$V(x,T) = \sup_{v\in\mathcal{L}_2[0,r]} \left\{ \int_0^r \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds + V(x(r), T-r) : x(0) = x \right\} \tag{2.40}$$

for all $r \in [0,T]$.

Proof: Fix $r \in [0,T]$ and define the switching disturbance

$$\hat{v}(s) = \begin{cases} v(s) & s \leq r, \\ \tilde{v}(s) & s > r, \end{cases}$$

where $v \in \mathcal{L}_2[0,r]$ and $\tilde{v} \in \mathcal{L}_2[r,T]$. Disturbance $\hat{v}$ is suboptimal in the definition of the finite horizon value function $V(x,T)$ (2.33), so that

$$V(x,T) \geq \int_0^r \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds + \int_r^T \left[c(\tilde{x}(s)) - \gamma^2|\tilde{v}(s)|^2\right] ds. \tag{2.41}$$

With $v \in \mathcal{L}_2[0,r]$ fixed, (2.41) holds for any $\tilde{v} \in \mathcal{L}_2[r,T]$. Hence, taking the supremum over $\tilde{v} \in \mathcal{L}_2[r,T]$ of both sides and applying the definition of $V(x,T)$ (2.33),

$$\begin{aligned} V(x,T) &\geq \int_0^r \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds + \sup_{\tilde{v}\in\mathcal{L}_2[r,T]} \left\{ \int_r^T \left[c(\tilde{x}(s)) - \gamma^2|\tilde{v}(s)|^2\right] ds \right\} \\ &= \int_0^r \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds + V(x(r), T-r). \end{aligned} \tag{2.42}$$

But, (2.42) holds for any $v \in \mathcal{L}_2[0,r]$. Hence, taking the supremum over $v \in \mathcal{L}_2[0,r]$ (and noting that $x(0) = x$),

$$V(x,T) \geq \sup_{v\in\mathcal{L}_2[0,r]} \left\{ \int_0^r \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds + V(x(r), T-r) : x(0) = x \right\}, \tag{2.43}$$

which proves inequality in one direction. To prove the other direction, let $v^\delta \in \mathcal{L}_2[0,T]$ be a δ-optimal disturbance for $V(x,T)$. That is (from (2.33)), given $\delta > 0$,

$$\begin{aligned} V(x,T) - \delta &< \int_0^T \left[c(x^\delta(s)) - \gamma^2|v^\delta(s)|^2\right] ds \\ &= \int_0^r \left[c(x^\delta(s)) - \gamma^2|v^\delta(s)|^2\right] ds + \int_r^T \left[c(\tilde{x}^\delta(s)) - \gamma^2|\tilde{v}^\delta(s)|^2\right] ds, \end{aligned}$$

where v^δ is labelled as $\tilde{v}^\delta$ on the interval $[r,T]$. Treating $v^\delta \in \mathcal{L}_2[0,r]$ and $\tilde{v}^\delta \in \mathcal{L}_2[0,r]$ as independent, noting first that $\tilde{v}^\delta$ is suboptimal on the interval $[r,T]$, and second that

v^δ is suboptimal on the interval $[0, r]$,

$$\begin{aligned} V(x,T) - \delta &\leq \int_0^r \left[c(x^\delta(s)) - \gamma^2 |v^\delta(s)|^2 \right] ds + \sup_{\tilde{v} \in \mathcal{L}_2[r,T]} \left\{ \int_r^T \left[c(\tilde{x}(s)) - \gamma^2 |\tilde{v}|^2 \right] ds \right\} \\ &= \int_0^r \left[c(x^\delta(s)) - \gamma^2 |v^\delta(s)|^2 \right] ds + V(x^\delta(r), T - r) \\ &\leq \sup_{v \in \mathcal{L}_2[0,r]} \left\{ \int_0^r \left[c(x(s)) - \gamma^2 |v(s)|^2 \right] ds + V(x(r), T-r) : x(0) = x \right\}, \end{aligned} \tag{2.44}$$

which holds for every $\delta > 0$. Choosing $\delta > 0$ arbitrarily small in (2.44) and combining with the opposite inequality (2.43) yields the dynamic programming equation (2.40).

■

In the next section, the behaviour of trajectories corresponding to near optimal disturbances for $V(x,T)$ (2.33) is considered. A useful corollary following from the dynamic programming principle is that a near optimal disturbance $v^\delta \in \mathcal{L}_2[0,T]$ for initial state x and horizon T is also near optimal for initial state $x^\delta(r)$ and horizon $T - r$, where $x^\delta(\cdot)$ is the trajectory corresponding to v^δ. That is, near optimal disturbances remain near optimal along the corresponding trajectory.

Corollary 2.5.9 *Suppose that $v^\delta \in \mathcal{L}_2[0,T]$ is δ-optimal for the finite horizon value function $V(x,T)$ (2.33). For any fixed $r \in [0,T]$, define $\bar{v}^\delta \in \mathcal{L}_2[0,T-r]$,*

$$\bar{v}^\delta(s) = v^\delta(s + r). \tag{2.45}$$

Then, $\bar{v}^\delta \in \mathcal{L}_2[0,T-r]$ is δ-optimal for $V(x^\delta(r), T-r)$, where $x^\delta(\cdot)$ is the trajectory corresponding to $v^\delta \in \mathcal{L}_2[0,T]$.

Proof: Since $v^\delta \in \mathcal{L}_2[0,T]$ is δ-optimal for $V(x,T)$ (2.33),

$$\begin{aligned} V(x,T) - \delta &\leq \int_0^T \left[c(x^\delta(s)) - \gamma^2 |v^\delta(s)|^2 \right] ds \\ &= \int_0^r \left[c(x^\delta(s)) - \gamma^2 |v^\delta(s)|^2 \right] ds + \int_0^{T-r} \left[c(\bar{x}^\delta(s)) - \gamma^2 |\bar{v}^\delta(s)|^2 \right] ds. \end{aligned}$$

where $r \in [0,T]$, $x^\delta(0) = x$, and $\bar{x}^\delta(0) = x^\delta(r)$. Rearranging,

$$V(x,T) - \int_0^r \left[c(x^\delta(s)) - \gamma^2 |v^\delta(s)|^2 \right] ds - \delta \leq \int_0^{T-r} \left[c(\bar{x}^\delta(s)) - \gamma^2 |\bar{v}^\delta(s)|^2 \right] ds. \tag{2.46}$$

But, applying Theorem 2.5.8, $V(x,T)$ (2.33) satisfies the dynamic programming equation (2.40). Furthermore, v^δ is suboptimal in (2.40), so that

$$V(x,T) \geq \int_0^r \left[c(x^\delta(s)) - \gamma^2|v^\delta(s)|^2\right] ds + V(x^\delta(r), T-r).$$

Rearranging,

$$V(x^\delta(r), T-r) \leq V(x,T) - \int_0^r \left[c(x^\delta(s)) - \gamma^2|v^\delta(s)|^2\right] ds. \tag{2.47}$$

Combining inequalities (2.46) and (2.47),

$$V(x^\delta(r), T-r) - \delta \leq \int_0^{T-r} \left[c(\bar{x}^\delta(s)) - \gamma^2|\bar{v}^\delta(s)|^2\right] ds. \tag{2.48}$$

But, from (2.33),

$$V(x^\delta(r), T-r) = \sup_{v\in\mathcal{L}_2[0,T-r]} \left\{\int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds : x(0) = x^\delta(r)\right\}. \tag{2.49}$$

Hence, from (2.48) and (2.49), $\bar{v}^\delta \in \mathcal{L}_2[0, T-r]$ must be δ-optimal for $V(x^\delta(r), T-r)$.

■

Recalling Theorem 2.5.4, explicit computation of the finite horizon value function $V(x,T)$ (2.33) is useful for testing if a system exhibits power gain. With $V(x,T)$ defined as the value function of a variational problem, dynamic programming allows the reformulation of this variational problem as a dynamic programming principle (2.40). However, such a dynamic programming principle can be interpreted as the integral representation of a partial differential equation, as stated in the following theorem [17].

Theorem 2.5.10 *Suppose that the finite horizon value function $V(x,T)$ (2.33) is continuous. Then, $V(x,T)$ (2.33) is a viscosity solution of the nonstationary partial differential equation (PDE)*

$$0 = -\frac{\partial V}{\partial T}(x,T) + H(x, \nabla_x V(x,T)), \tag{2.50}$$

where $H(x,p)$ is the Hamiltonian

$$H(x,p) = \sup_{v\in\mathbf{R}^p} \left\{p \cdot [a(x) + b(x)v] + c(x) - \gamma^2|v|^2\right\}. \tag{2.51}$$

With Theorem 2.5.10, computation of a solution of the variational problem defined by $V(x,T)$ (2.33) now involves the computation of a solution to the nonstationary PDE (2.50). Although analytical solutions are rare, such PDEs are amenable to solution by approximation methods such as finite differences. This will be considered in detail in Chapter 4.

2.5.4 Behaviour of Near Optimal Trajectories

From Definition 2.5.1, the infinite horizon value function $V(x,T)$ (2.33) may be interpreted as a measure of the most energy that can be retrieved from a system starting in initial state x using disturbances defined on the finite interval $[0,T]$. Consequently, the optimal disturbance for $V(x,T)$ is of considerable interest, since it is this disturbance which excites the dynamics of a system yielding maximal energy generation on the interval $[0,T]$. Given that the optimal disturbance is in this sense "worst case", the stability of a system in the presence of the optimal disturbance is also of concern. (Note that since the worst case disturbance may not exist, near optimal disturbances must be considered.)

In this section, we begin by obtaining square integral bounds on near optimal disturbances (and corresponding outputs) for fixed initial state x, horizon T, and gain γ. Using these bounds, a form of stability of near optimal trajectories is proved. This form of stability guarentees that the final state $x^\delta(T)$ corresponding to any near optimal trajectory $v^\delta \in \mathcal{L}_2[0,T]$ is bounded uniformly with respect to T and uniformly with respect to x on compact sets. This uniformity property will then be used in later sections to prove properties of the worst case power generation of systems, and in determining the behaviour of the trajectory corresponding to the "worst case" disturbance defined on an infinite time horizon.

Proposition 2.5.11 *Suppose that $\gamma > \hat{\gamma}$, and $V_{\hat{\gamma}}(x,T) < \infty$, where $V_{\hat{\gamma}}(x,T)$ is the finite horizon value function $V(x,T)$ (2.33) for gain $\hat{\gamma}$. Given $\delta > 0$, let $v_{x,T}^{\gamma,\delta} \in \mathcal{L}_2[0,T]$ be δ-optimal in the definition of $V_\gamma(x,T)$ for gain γ, initial state x, and horizon T. Then,*

$$\int_0^T |v_{x,T}^{\gamma,\delta}(s)|^2\, ds \;\leq\; \frac{V_{\hat{\gamma}}(x,T) - V_\gamma(x,T)}{\gamma^2 - \hat{\gamma}^2} + \frac{\delta}{\gamma^2 - \hat{\gamma}^2}. \tag{2.52}$$

Proof: By Proposition 2.5.3, $V_\gamma(x,T) \leq V_{\hat{\gamma}}(x,T) < \infty$. That is, $V_\gamma(x,T)$ is finite. With $v_{x,T}^{\gamma,\delta} \in \mathcal{L}_2[0,T]$ δ-optimal for gain γ, initial state x, and horizon T,

$$\int_0^T \left[c(x_{x,T}^{\gamma,\delta}(s)) - \gamma^2 |v_{x,T}^{\gamma,\delta}(s)|^2\right] ds \;\geq\; V_\gamma(x,T) - \delta,$$

where $x_{x,T}^{\gamma,\delta}(\cdot)$ is the trajectory corresponding to $v_{x,T}^{\gamma,\delta}(\cdot)$. Rearranging yields that

$$\gamma^2 \int_0^T |v_{x,T}^{\gamma,\delta}(s)|^2\, ds \;\leq\; \int_0^T c(x_{x,T}^{\gamma,\delta}(s))\, ds - V_\gamma(x,T) + \delta. \tag{2.53}$$

However, $v_{x,T}^{\gamma,\delta}(\cdot)$ is suboptimal for gain $\hat{\gamma}$, so that

$$\int_0^T \left[c(x_{x,T}^{\gamma,\delta}(s)) - \hat{\gamma}^2 |v_{x,T}^{\gamma,\delta}(s)|^2\right] ds \leq V_{\hat{\gamma}}(x,T). \tag{2.54}$$

Rearranging (2.54) and combining with (2.53),

$$\left(\gamma^2 - \hat{\gamma}^2\right) \int_0^T |v_{x,T}^{\gamma,\delta}(s)|^2 ds \leq V_{\hat{\gamma}}(x,T) - V_\gamma(x,T) + \delta$$

Noting that $\gamma > \hat{\gamma}$ proves the assertion. ∎

A similar bound can be proved for the accumulated unpenalized running cost along a near optimal trajectory.

Proposition 2.5.12 *Suppose that $\gamma > \hat{\gamma}$, and $V_{\hat{\gamma}}(x,T) < \infty$, where $V_{\hat{\gamma}}(x,T)$ is the finite horizon value function $V(x,T)$ for gain $\hat{\gamma}$. Given $\delta > 0$, let $v_{x,T}^{\gamma,\delta} \in \mathcal{L}_2[0,T]$ be δ-optimal in the definition of $V_\gamma(x,T)$ for gain γ, initial state x, and horizon T. Then,*

$$\int_0^T c(x_{x,T}^{\gamma,\delta}(s))\, ds \leq \frac{\gamma^2 V_{\hat{\gamma}}(x,T) - \hat{\gamma}^2 V_\gamma(x,T)}{\gamma^2 - \hat{\gamma}^2} + \frac{\hat{\gamma}^2 \delta}{\gamma^2 - \hat{\gamma}^2}. \tag{2.55}$$

Proof: Since $v_{x,T}^{\gamma,\delta}$ is suboptimal in the definition of $V_{\hat{\gamma}}(x,T)$,

$$\int_0^T c(x_{x,T}^{\gamma,\delta}(s))\, ds \leq V_{\hat{\gamma}}(x,T) + \hat{\gamma}^2 \int_0^T |v_{x,T}^{\gamma,\delta}(s)|^2 ds.$$

Applying the bound (2.52) for the energy of $v_{x,T}^{\gamma,\delta}$ implies that

$$\int_0^T c(x_{x,T}^{\gamma,\delta}(s))\, ds \leq V_{\hat{\gamma}}(x,T) + \frac{\hat{\gamma}^2}{\gamma^2 - \hat{\gamma}^2}\left(V_{\hat{\gamma}}(x,T) - V_\gamma(x,T) + \delta\right),$$

which gives the desired result. ∎

Using similar arguments, square integral lower bounds for both the near optimal disturbance and the unpenalized running cost can be obtained.

Proposition 2.5.13 *Suppose that $\tilde{\gamma} > \gamma > \hat{\gamma}$, and $V_{\hat{\gamma}}(x,T) < \infty$, where $V_{\hat{\gamma}}(x,T)$ is the finite horizon value function $V(x,T)$ (2.33) for gain $\hat{\gamma}$. Given $\delta > 0$, let $v_{x,T}^{\gamma,\delta} \in \mathcal{L}_2[0,T]$ be δ-optimal in the definition of $V_\gamma(x,T)$ for gain γ, initial state x, and horizon T. Then,*

$$\int_0^T |v_{x,T}^{\gamma,\delta}(s)|^2 ds \geq \frac{V_\gamma(x,T) - V_{\tilde{\gamma}}(x,T)}{\tilde{\gamma}^2 - \gamma^2} - \frac{\delta}{\tilde{\gamma}^2 - \gamma^2} \tag{2.56}$$

Proof: With $v_{x,T}^{\gamma,\delta} \in \mathcal{L}_2[0,T]$ δ-optimal for gain γ, initial state x, and horizon T,

$$\int_0^T \left[c(x_{x,T}^{\gamma,\delta}(s)) - \gamma^2 |v_{x,T}^{\gamma,\delta}(s)|^2\right] ds \geq V_\gamma(x,T) - \delta$$

$$\Rightarrow \int_0^T c(x_{x,T}^{\gamma,\delta}(s))\, ds \geq V_\gamma(x,T) - \delta + \gamma^2 \int_0^T |v_{x,T}^{\gamma,\delta}(s)|^2 ds, \tag{2.57}$$

where $x_{x,T}^{\gamma,\delta}(\cdot)$ is the corresponding trajectory. But, $v_{x,T}^{\gamma,\delta}$ is suboptimal for gain $\tilde{\gamma}$, so that

$$\begin{aligned} V_{\tilde{\gamma}}(x,T) &\geq \int_0^T \left[c(x_{x,T}^{\gamma,\delta}(s)) - \tilde{\gamma}^2 |v_{x,T}^{\gamma,\delta}(s)|^2\right] ds \\ \Rightarrow \tilde{\gamma}^2 \int_0^T |v_{x,T}^{\gamma,\delta}(s)|^2 \, ds &\geq \int_0^T c(x_{x,T}^{\gamma,\delta}(s)) - V_{\tilde{\gamma}}(x,T). \end{aligned} \tag{2.58}$$

Combining (2.57) and (2.58),

$$(\tilde{\gamma}^2 - \gamma^2) \int_0^T |v_{x,T}^{\gamma,\delta}(s)|^2 \, ds \geq V_\gamma(x,T) - V_{\tilde{\gamma}}(x,T) - \delta.$$

Noting that $\tilde{\gamma} > \gamma$ proves the assertion. ∎

Proposition 2.5.14 *Suppose that $\hat{\gamma} > \gamma > \tilde{\gamma}$, and $V_{\hat{\gamma}}(x,T) < \infty$, where $V_{\hat{\gamma}}(x,T)$ is the finite horizon value function $V(x,T)$ (2.33) for gain $\hat{\gamma}$. Given $\delta > 0$, let $v_{x,T}^{\gamma,\delta} \in \mathcal{L}_2[0,T]$ be δ-optimal in the definition of $V_\gamma(x,T)$ for gain γ, initial state x, and horizon T. Then,*

$$\int_0^T c(x_{x,T}^{\gamma,\delta}(s)) \, ds \geq \frac{\tilde{\gamma}^2 V_\gamma(x,T) - \gamma^2 V_{\tilde{\gamma}}(x,T)}{\tilde{\gamma}^2 - \gamma^2} - \frac{\tilde{\gamma}^2 \delta}{\tilde{\gamma}^2 - \gamma^2}. \tag{2.59}$$

Proof: Combine (2.57) and (2.58). ∎

The upper bounds provided by Propositions 2.5.11 and 2.5.12 both involve the finite horizon value function $V(x,T)$ (2.33) for gains γ and $\hat{\gamma}$, where $\gamma > \hat{\gamma}$. However, by applying stability assumption (A7), boundedness assumption (A10), and growth assumption (A12), Theorem 2.5.6 may be applied to obtain more useful estimates for the energy of the disturbance and output.

Proposition 2.5.15 *Suppose that assumptions (A7), (A10), and (A12) hold. Then, there exists a gain $\bar{\gamma} \geq 0$ and nonnegative constants B_5, B_6, B_7, B_8, B_9, B_{10} such that*

$$\int_0^T |v_{x,T}^{\gamma,\delta}(s)|^2 \, ds \leq B_5 T + B_6 |x|^2 + B_7 \delta, \tag{2.60}$$

$$\int_0^T c(x_{x,T}^{\gamma,\delta}(s)) \, ds \leq B_8 T + B_9 |x|^2 + B_{10} \delta, \tag{2.61}$$

for all gains $\gamma > \bar{\gamma}$, where $v_{x,T}^{\gamma,\delta}(\cdot)$ and $x_{x,T}^{\gamma,\delta}(\cdot)$ are the δ-optimal disturbance and trajectory, respectively, in the definition of $V_\gamma(x,T)$ for gain γ, initial state x, and horizon T.

Proof: Applying Theorem 2.5.6, there exists a gain $\bar{\gamma} \geq 0$ and constants B_3 and B_4

such that

$$V_{\bar{\gamma}}(x,T) \leq B_3T + B_4|x|^2.$$

Furthermore, by definition of $V_\gamma(x,T)$ (2.33), $V_\gamma(x,T) \geq 0$ for all $x \in \mathbf{R}^n$, $T \geq 0$. Hence, for any $\gamma > \bar{\gamma}$, Proposition 2.5.11 implies that

$$\begin{aligned}
\int_0^T |v_{x,T}^{\gamma,\delta}(s)|^2\, ds &\leq \frac{V_{\bar{\gamma}}(x,T)}{\gamma^2-\bar{\gamma}^2} - \frac{V_\gamma(x,T)}{\gamma^2-\bar{\gamma}^2} + \frac{\delta}{\gamma^2-\bar{\gamma}^2} \\
&\leq \frac{V_{\bar{\gamma}}(x,T)}{\gamma^2-\bar{\gamma}^2} + \frac{\delta}{\gamma^2-\bar{\gamma}^2} \\
&\leq \frac{B_3T + B_4|x|^2}{\gamma^2-\bar{\gamma}^2} + \frac{\delta}{\gamma^2-\bar{\gamma}^2} \\
&= B_5T + B_6|x|^2 + B_7\delta,
\end{aligned}$$

which is (2.60), where $B_5 = \frac{B_3}{\gamma^2-\bar{\gamma}^2}$, $B_6 = \frac{B_4}{\gamma^2-\bar{\gamma}^2}$, and $B_7 = \frac{1}{\gamma^2-\bar{\gamma}^2}$. Similarly, applying Proposition 2.5.12,

$$\begin{aligned}
\int_0^T c(x_{x,T}^{\gamma,\delta}(s))\, ds &\leq \frac{\gamma^2 V_{\bar{\gamma}}(x,T)}{\gamma^2-\bar{\gamma}^2} + \frac{\bar{\gamma}^2\delta}{\gamma^2-\bar{\gamma}^2} \\
&\leq \frac{\gamma^2 B_3T + \gamma^2 B_4|x|^2}{\gamma^2-\bar{\gamma}^2} + \frac{\bar{\gamma}^2\delta}{\gamma^2-\bar{\gamma}^2} \\
&= \gamma^2B_5T + \gamma^2B_6|x|^2 + \bar{\gamma}^2B_7\delta \\
&= B_8T + B_9|x|^2 + B_{10}\delta,
\end{aligned}$$

which is (2.61), where $B_8 = \gamma^2B_5$, $B_9 = \gamma^2B_6$, and $B_{10} = \bar{\gamma}^2B_7$. ∎

With these growth bounds on the δ-optimal disturbance $v_{x,T}^{\gamma,\delta}$ and the corresponding δ-optimal output $z_{x,T}^{\gamma,\delta}(s) = c(x_{x,T}^{\gamma,\delta}(s))$, the aim now is to prove via the detectability assumptions (A14), (A15) that the final state $x_{x,T}^{\gamma,\delta}(T)$ for any δ-optimal trajectory $x_{x,T}^{\gamma,\delta}(s)$ is confined uniformly with respect to T and x to a compact set for all initial states x in a compact set.

Define the sets

$$B_R = \{x \in \mathbf{R}^n : |x| \leq R\}, \tag{2.62}$$

$$B_\rho = \{x \in \mathbf{R}^n : |x| \leq \rho\}, \tag{2.63}$$

$$K_{3B_8} = \{x \in \mathbf{R}^n : c(x) \leq 3B_8\}, \tag{2.64}$$

where B_8 is coefficient of T in (2.61). By assumptions (A14) and (A15), there exists a $\rho < \infty$ such that $K_{3B_8} \subset B_\rho$. Defining

$$\bar{\rho} = \max_{x \in K_{3B_8}} \{|x|\}, \tag{2.65}$$

then $\rho > \bar{\rho}$. Furthermore, with $R > \rho$,

$$K_{3B_8} \subset B_\rho \subset B_R, \tag{2.66}$$

as shown in Figure 2.4.

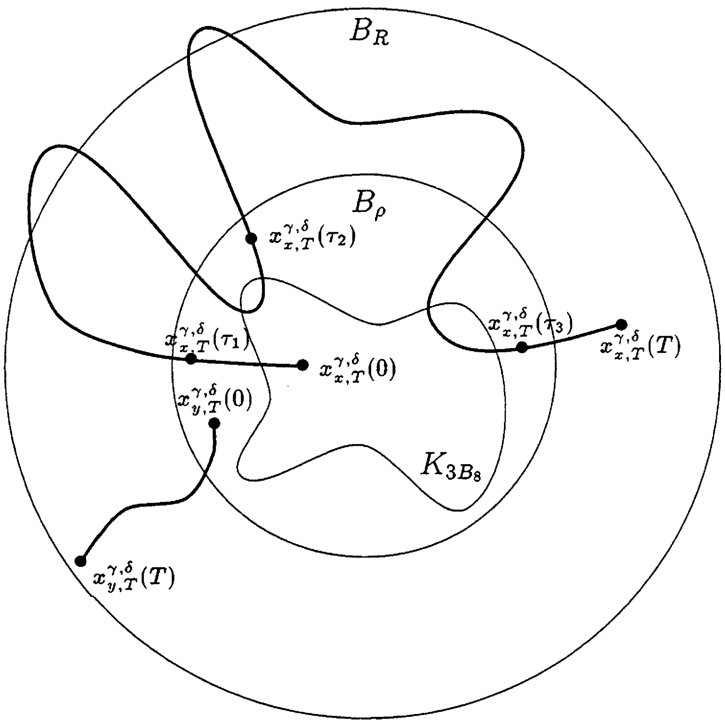


Figure 2.4: Set Ordering $K_{3B_8} \subset B_\rho \subset B_R$

Using the various estimates obtained thus far, it is possible to show that the final state of any near optimal trajectory on $[0, T]$ must be contained in B_R, for a suitably large choice of R, provided that the horizon T satisfies the inequality $T \leq \bar{T}$, for some $\bar{T} < \infty$. This is intuitively clear, given that trajectories can only move so far in a finite time, as is demonstrated in the following lemma.

Lemma 2.5.16 *Suppose that assumptions (A7) and (A10) hold. Let $x_{x,T}^{\gamma,\delta}(\cdot)$ be the trajectory with initial state $x \in B_\rho$ corresponding to a δ-optimal disturbance $v_{x,T}^{\gamma,\delta} \in \mathcal{L}_2[0,T]$, $\delta \leq \bar{\delta} < \infty$, and suppose that $T \leq \bar{T}$ for some $\bar{T} < \infty$. Then, there exists an $0 \leq R_\rho < \infty$ given by*

$$R_\rho = \sqrt{(1 + B_2 B_6)\rho^2 + (B_1 + B_2 B_5)\bar{T} + B_2 B_7 \bar{\delta}}, \tag{2.67}$$

such that the trajectory $x_{x,T}^{\gamma,\delta}(s)$ is confined to the ball B_{R_ρ} for all $s \in [0, T]$.

Proof: Applying assumptions (A7) and (A10), Lemma 2.5.5 implies that there exists constants B_1 and B_2 such that

$$|x(t)|^2 \leq |x|^2 + B_1 t + B_2 \int_0^t |v(s)|^2 \, ds$$

for all $t \in [0,T]$, where $v(\cdot)$ is any disturbance in $\mathcal{L}_2[0,T]$. So, since $v_{x,T}^{\gamma,\delta} \in \mathcal{L}_2[0,T]$, $x \in B_\rho$, and $0 \leq t \leq T \leq \bar{T}$,

$$|x_{x,T}^{\gamma,\delta}(t)|^2 \leq \rho^2 + B_1 \bar{T} + B_2 \int_0^T |v_{x,T}^{\gamma,\delta}(s)|^2 \, ds.$$

But, applying Proposition 2.5.15, (2.60),

$$\begin{aligned} |x_{x,T}^{\gamma,\delta}(t)|^2 &\leq \rho^2 + B_1 \bar{T} + B_2 \left(B_5 T + B_6 |x|^2 + B_7 \delta \right) \\ &\leq (1 + B_2 B_6)\rho^2 + (B_1 + B_2 B_5)\bar{T} + B_2 B_7 \bar{\delta}. \end{aligned}$$

Defining $R_\rho = \sqrt{(1 + B_2 B_6)\rho^2 + (B_1 + B_2 B_5)\bar{T} + B_2 B_7 \bar{\delta}}$ completes the proof. ∎

Lemma 2.5.16 demonstrates that a uniform bound on the excursions of near optimal trajectories exists provided that the time horizon is also uniformly bounded. Without the uniform bound on the time horizon, the corresponding result is more difficult to prove. However, it is possible to demonstrate that the the final state of any near optimal trajectory is uniformly bounded, for any choice of time horizon, and any choice of initial state in a predefined compact set. As a first step towards this result, the following lemma demonstrates that a near optimal trajectory must visit such a compact set at least once, given a sufficiently large time horizon.

Lemma 2.5.17 *Suppose that assumptions (A7), (A10), and (A12) hold. Suppose also that $T > \bar{T}$, where*

$$\bar{T} = \frac{B_9 \rho^2 + B_{10} \bar{\delta}}{B_8}. \tag{2.68}$$

Then, there exists an $s \in [0,T]$ such that $x_{x,T}^{\gamma,\delta}(s) \in K_{3B_8}$, where $x_{x,T}^{\gamma,\delta}(\cdot)$ is the trajectory with initial state $x \in B_\rho$ corresponding to a δ-optimal disturbance for the finite horizon value function $V(x,T)$ (2.33).

Proof: Suppose that $x_{x,T}^{\gamma,\delta}(s) \notin K_{3B_8}$ for all $s \in [0,T]$. Then, from (2.64), this implies that

$$c(x_{x,T}^{\gamma,\delta}(s)) > 3B_8$$

for all $s \in [0,T]$. Hence, given that $T \geq \bar{T}$ (2.68),

$$\begin{aligned}\int_0^T c(x_{x,T}^{\gamma,\delta}(s))\, ds &> 3B_8 T \\ &> 2B_8 + B_8\bar{T} \\ &= 2B_8 + B_9\rho^2 + B_{10}\bar{\delta} \\ &\geq 2B_8 + B_9\rho^2 + B_{10}\delta. \end{aligned} \tag{2.69}$$

But, applying Proposition 2.5.15, (2.61),

$$B_8 T + B_9|x|^2 + B_{10}\delta \;\geq\; \int_0^T c(x_{x,T}^{\gamma,\delta}(s))\, ds. \tag{2.70}$$

Since the initial state is $x \in B_\rho$, $|x| \leq \rho$. Hence, combining (2.69) and (2.70),

$$\int_0^T c(x_{x,T}^{\gamma,\delta}(s))\, ds \;>\; B_8 T + \int_0^T c(x_{x,T}^{\gamma,\delta}(s))\, ds,$$

which is a contradiction, since $T > \bar{T} \geq 0$. That is, there exists an $s \in [0,T]$ such that $x_{x,T}^{\gamma,\delta}(s) \in K_{3B_8}$. ■

Although Lemma 2.5.17 establishes that any δ-optimal trajectory must enter the set K_{3B_8}, there is no information about when this entry may occur, except that it must be in the interval $[0,T]$. Consequently, it would seem possible that the trajectory may enter and exit K_{3B_8} very early in the interval $[0,T]$, allowing scope for the trajectory to escape a predefined compact set for the remainder of the interval. However, the following corollary demonstrates that if the exit from K_{3B_8} occurs sufficiently early in the interval $[0,T]$, then the trajectory must again enter K_{3B_8} at some later time, as per Figure 2.4.

Corollary 2.5.18 *Suppose that assumptions (A7), (A10), and (A12) hold. Let $x_{x,T}^{\gamma,\delta}(\cdot)$ be the trajectory with initial state $x \in B_\rho$ corresponding to a δ-optimal disturbance for the finite horizon value function $V(x,T)$ (2.33). Suppose that there exists a $\tau \in [0,T]$ such that $x_{x,T}^{\gamma,\delta}(\tau) \in B_\rho \setminus K_{3B_8}$ and $T - \tau > \bar{T}$, where $\bar{T}$ is given by (2.68). Then, there exists an $s_1 \in (\tau, T]$ such that $x_{x,T}^{\gamma,\delta}(s_1) \in K_{3B_8}$.*

Proof: Since disturbance $v_{x,T}^{\gamma,\delta} \in \mathcal{L}_2[0,T]$ is δ-optimal for $V(x,T)$, Corollary 2.5.9 implies that the disturbance $\bar{v}^\delta \in \mathcal{L}_2[0,T-\tau]$ is δ-optimal for $V(x_{x,T}^{\gamma,\delta}(\tau), T-\tau)$, where

$$\bar{v}^\delta(s) \;=\; v_{x,T}^{\gamma,\delta}(s+\tau),$$

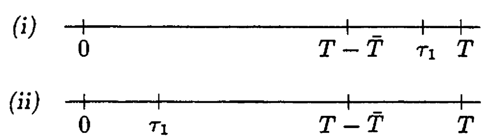
and $\bar{x}^\delta(\cdot)$ is the corresponding trajectory on $[0, T-\tau]$. But, $\bar{x}^\delta(0) = x_{x,T}^{\gamma,\delta}(\tau) \in B_\rho$ and, by assumption, $T - \tau > \bar{T}$. Hence, applying Lemma 2.5.17 and noting that

$\bar{x}^{\delta}(0) \notin K_{3B_8}$, there exists an $s \in (0, T-\tau]$ such that $\bar{x}^{\delta}(s) \in K_{3B_8}$. But, by definition, $x_{x,T}^{\gamma,\delta}(s+\tau) = \bar{x}^{\delta}(s)$. Hence, there exists an $s_1 = s + \tau$ such that $x_{x,T}^{\gamma,\delta}(s_1) \in K_{3B_8}$. ■

Using Corollary 2.5.18, it is now possible to show that a near optimal trajectory must at least visit the set B_ρ in the time interval $[T - \bar{T}, T]$.

Lemma 2.5.19 *Suppose that assumptions (A7), (A10), and (A12) hold. Let* $T > \bar{T}$*, where* $\bar{T}$ *is given by (2.68), and let* $x_{x,T}^{\gamma,\delta}(\cdot)$ *be the trajectory with initial state* $x \in B_\rho$ *corresponding to a* δ*-optimal disturbance for the finite horizon value function* $V(x, T)$ *(2.33). Then, there exists an* $s_1 \in [T - \bar{T}, T]$ *such that* $x_{x,T}^{\gamma,\delta}(s_1) \in B_\rho$.

Proof: Let $x_{x,T}^{\gamma,\delta}(\cdot)$ be the trajectory with initial state $x_{x,T}^{\gamma,\delta}(0) = x \in B_\rho$ corresponding to a δ-optimal disturbance $v_{x,T}^{\gamma,\delta} \in \mathcal{L}_2[0, T]$. Then, either $x \in K_{3B_8}$ and $x_{x,T}^{\gamma,\delta}(\cdot)$ never exits K_{3B_8}, or there exists a $\tau_1 \in [0, T]$ such that $x_{x,T}^{\gamma,\delta}(\tau_1) \in B_\rho \setminus K_{3B_8}$. If the former holds, then the proof is complete. So, without loss of generality, assume there exists a $\tau_1 \in [0, T]$ such that $x_{x,T}^{\gamma,\delta}(\tau_1) \in B_\rho \setminus K_{3B_8}$. Construct two cases for the location of τ_1 in the interval $[0, T]$:



Case (i), $\tau_1 \in [T - \bar{T}, T]$: By definition of τ_1, $x_{x,T}^{\gamma,\delta}(\tau_1) \in B_\rho$, so the proof for this case is complete.

Case (ii), $\tau_1 \in [0, T - \bar{T})$: Since $x \in B_\rho$, $x_{x,T}^{\gamma,\delta}(\tau_1) \in B_\rho \setminus K_{3B_8}$, and $T - \tau_1 > \bar{T}$ (by definition of *Case (ii)*), Corollary 2.5.18 implies that there exists an $s_2 \in (\tau_1, T]$ such that $x_{x,T}^{\gamma,\delta}(s_2) \in K_{3B_8}$. Suppose that $s_2 \in [T - \bar{T}, T]$. Then, since $K_{3B_8} \subset B_\rho$, the proof is complete. Alternatively, suppose that $s_2 \notin [T - \bar{T}, T]$. Then, by continuity of $x_{x,T}^{\gamma,\delta}(\cdot)$, there exists a τ_2 such that $x_{x,T}^{\gamma,\delta}(\tau_2) \in B_\rho \setminus K_{3B_8}$. That is, τ_2 may be considered in terms of *Case (i)* or *Case (ii)* above. Applying the above reasoning iteratively defines an increasing sequence $\{\tau_k\}$, $\tau_k \in (\tau_{k-1}, T]$, $\tau_1 \in [0, T]$, which terminates with $\tau_n \in [T - \bar{T}, T]$ for some integer $n \geq 1$. Note that by definition, $x_{x,T}^{\gamma,\delta}(\tau_k) \in B_\rho$. Hence, there exists an $s_1 = \tau_n \in [T - \bar{T}, T]$ such that $x_{x,T}^{\gamma,\delta}(s_1) \in B_\rho$. ■

Lemmas 2.5.16 and 2.5.19 may now be combined to show that the final state of a near optimal trajectory is uniformly bounded, for any choice of horizon and any choice of initial state within a predefined compact set.

Theorem 2.5.20 *Suppose that assumptions (A7), (A10), and (A12) hold. Let B_ρ (2.63) be the ball of radius $\rho < \infty$, where $\rho > \bar{\rho}$ and $\bar{\rho}$ is given by (2.65). That is, let*

$$\begin{aligned} B_\rho &= \{x \in \mathbf{R}^n : |x| \leq \rho\}, \quad \rho > \bar{\rho}, \\ \bar{\rho} &= \max_{x \in K_{3B_8}} \{|x|\}. \end{aligned}$$

Let B_{R_ρ} (2.62) be the ball of radius $R_\rho < \infty$ (2.67). That is,

$$\begin{aligned} B_{R_\rho} &= \{x \in \mathbf{R}^n : |x| \leq R_\rho\}, \\ R_\rho &= \sqrt{(1 + B_2 B_6)\rho^2 + (B_1 + B_2 B_5)\bar{T} + B_2 B_7 \bar{\delta}}, \\ \bar{T} &= \frac{B_9 \rho^2 + B_{10}\bar{\delta}}{B_8}. \end{aligned}$$

Let $x_{x,T}^{\gamma,\delta}(\cdot)$ be the trajectory corresponding to a δ-optimal disturbance $v_{x,T}^{\gamma,\delta} \in \mathcal{L}_2[0,T]$, $\delta \leq \bar{\delta} < \infty$, for the finite horizon value function $V(x,T)$ (2.33). Then, if the initial state $x_{x,T}^{\gamma,\delta}(0) \in B_\rho$, then the trajectory $x_{x,T}^{\gamma,\delta}(\cdot)$ always returns to the compact set B_{R_ρ}, for any time horizon $T < \infty$. That is,

$$x \in B_\rho \quad \Rightarrow \quad x_{x,T}^{\gamma,\delta}(T) \in B_{R_\rho} \tag{2.71}$$

for any $T < \infty$.

Proof: Suppose that $T \leq \bar{T}$. Then, by Lemma 2.5.16, $x_{x,T}^{\gamma,\delta}(s) \in B_{R_\rho}$ for all $s \in [0,T]$. Hence, $x_{x,T}^{\gamma,\delta}(T) \in B_{R_\rho}$ completing the proof.

Alternatively, suppose that $T > \bar{T}$. Then, by Lemma 2.5.19, there exists an $s_1 \in [T - \bar{T}, T]$ such that $x_{x,T}^{\gamma,\delta}(s_1) \in B_\rho$. But, by Corollary 2.5.9, the disturbance

$$\bar{v}^\delta(s) \quad = \quad v_{x,T}^{\gamma,\delta}(s + s_1)$$

is δ-optimal for the finite horizon value function $V(x_{x,T}^{\gamma,\delta}(s_1), T - s_1)$. Let $\bar{x}^\delta(\cdot)$ denote the corresponding trajectory. Since $T - s_1 \leq \bar{T}$ and $\bar{x}^\delta(0) \in B_\rho$, Lemma 2.5.16 implies that $\bar{x}^\delta(T - s_1) \in B_{R_\rho}$. But, $\bar{x}^\delta(T - s_1) = x_{x,T}^{\gamma,\delta}(T)$, completing the proof. ∎

Remark 2.5.21 Since the above results apply for any δ-optimal disturbance where $0 < \delta \leq \bar{\delta} < \infty$, clearly these results must also hold for the optimal disturbance if it exists. This follows by taking $\delta \downarrow 0$. ◀

One application of Theorem 2.5.20 is in interpreting the initial value problem defined by the finite horizon value function $V(x,T)$ (2.33) in terms of a two point (fixed initial and final state) problem. This will be discussed in the next section.

2.5.5 Two Point Boundary Value Problem Interpretation

The finite horizon value function $V(x,T)$ (2.33) is defined as a calculus of variations problem with a differential equation constraint (given by the state equation (2.1)) which has a fixed initial state $x \in \mathbf{R}^n$, fixed final time $0 \le T < \infty$, and free final state [28]. A closely related problem is that defined by the following value function $V^f(x,\xi,T)$.

Definition 2.5.22 *Define the finite horizon fixed final state value function*

$$V^f(x,\xi,T) = \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right] \, ds : x(0) = x, x(T) = \xi \right\}. \quad (2.72)$$

Comparing the definitions of $V(x,T)$ (2.33) and $V^f(x,\xi,T)$ (2.72), clearly the difference between the two problems is that the value function $V^f(x,\xi,T)$ has an additional constraint. That is, the final state of the optimal trajectory is fixed rather than free. Consequently, intuition may lead us to the conclusion that $V(x,T)$ may be recovered by taking the supremum over the final state ξ of $V^f(x,\xi,T)$. This intuitive link between $V(x,T)$ and $V^f(x,\xi,T)$ will prove useful in developing the definition of required supply in Section 2.13.

Remark 2.5.23 If state ξ is unreachable from state x, the supremum in (2.72) is defined to be $-\infty$. That is,

$$V^f(x,\xi,T) = -\infty$$

if there does not exist a $v \in \mathcal{L}_2[0,T]$ such that $\varphi(T,0,x;v) = \xi$. ◀

Theorem 2.5.24 *The finite horizon value function $V(x,T)$ (2.33) is the supremum over all final states $\xi \in \mathbf{R}^n$ of the finite horizon fixed final state value function $V^f(x,\xi,T)$ (2.72). That is,*

$$V(x,T) = \sup_{\xi\in\mathbf{R}^n} \left\{ V^f(x,\xi,T) \right\} \quad (2.73)$$

for all $x \in \mathbf{R}^n$ and all $T \ge 0$.

Proof: Define the indicator function

$$\delta_\xi(x) = \begin{cases} -\infty & x \ne \xi, \\ 0 & x = \xi. \end{cases} \quad (2.74)$$

Then, directly from Definition 2.5.22, the value function $V^f(x,\xi,T)$ (2.72) may be rewritten as a free final state problem by encoding the final state constraint as a terminal cost which is $-\infty$ if the final state is not ξ. That is,

$$V^f(x,\xi,T) = \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right] \, ds + \delta_\xi(x(T)) : x(0) = x \right\}. \quad (2.75)$$

Using the fact that $\sup a - \sup b \le \sup(a-b)$, the definitions of $V^f(x,\xi,T)$ (2.75) and $V(x,T)$ (2.33) imply that

$$\begin{aligned} V^f(x,\xi,T) - V(x,T) &= \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right] \, ds + \delta_\xi(x(T)) : x(0) = x \right\} - \\ &\qquad \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right] \, ds : x(0) = x \right\} \\ &\le \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \delta_\xi(x(T)) : x(0) = x \right\} \\ &\le \sup_{x\in\mathbf{R}^n} \left\{ \delta_\xi(x) \right\} \\ &= 0 \end{aligned} \quad (2.76)$$

from (2.74). Since (2.76) holds for all $\xi \in \mathbf{R}^n$, clearly

$$\sup_{\xi\in\mathbf{R}^n} \left\{ V^f(x,\xi,T) \right\} \le V(x,T). \quad (2.77)$$

In order to prove the opposite inequality, let $v^\delta \in \mathcal{L}_2[0,T]$ be δ-optimal for $V(x,T)$ (2.33). That is,

$$\int_0^T \left[c(x^\delta(s)) - \gamma^2|v^\delta(s)|^2\right] \, ds \ \ge\ V(x,T) - \delta,$$

where $x^\delta(\cdot)$ is the trajectory with initial state $x^\delta(0) = x$ corresponding to the disturbance $v^\delta \in \mathcal{L}_2[0,T]$. Applying the definition of $\delta_\xi(\cdot)$ (2.74), it follows trivially that

$$\int_0^T \left[c(x^\delta(s)) - \gamma^2|v^\delta(s)|^2\right] \, ds + \delta_{x^\delta(T)}(x^\delta(T)) \ \ge\ V(x,T) - \delta. \quad (2.78)$$

But, $v^\delta(\cdot)$ is suboptimal in the definition of the finite horizon fixed final state value function $V(x, x^\delta(T), T)$. That is, applying (2.75),

$$\begin{aligned} V(x,x^\delta(T),T) &= \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right] \, ds + \delta_{x^\delta(T)}(x(T)) : x(0) = x \right\} \\ &\ge \int_0^T \left[c(x^\delta(s)) - \gamma^2|v^\delta(s)|^2\right] \, ds + \delta_{x^\delta(T)}(x^\delta(T)), \end{aligned} \quad (2.79)$$

where $x^\delta(0) = x$. So, combining inequalities (2.78) and (2.79),

$$V(x,x^\delta(T),T) \ \ge\ V(x,T) - \delta. \quad (2.80)$$

But, from inequality (2.77), $V(x,T) \geq \sup_{\xi \in \mathbf{R}^n} \left\{V^f(x,\xi,T)\right\} \geq V(x,x^\delta(T),T)$, so that

$$V(x,T) \ \geq \ \sup_{\xi \in \mathbf{R}^n} \left\{V^f(x,\xi,T)\right\} \ \geq \ V(x,T) - \delta,$$

for all $\delta > 0$, completing the proof. ∎

Combining Theorems 2.5.20 and 2.5.24, the following result demonstrates that the supremum in (2.73) is attained on a compact set provided that the choice of initial state is limited to a compact set.

Theorem 2.5.25 *Suppose that assumptions (A7), (A10), and (A12) hold. Then, given $\bar{\rho}$ (2.65), and any $\rho \in (\bar{\rho}, \infty)$, there exists a $R_\rho \in (0,\infty)$ (2.67) independent of x and T such that*

$$|x| \leq \rho \ \Rightarrow \ V(x,T) = \sup_{\xi \in B_{R_\rho}} \left\{V^f(x,\xi,T)\right\}, \tag{2.81}$$

where $B_{R_\rho} = \{\xi \in \mathbf{R}^n : |\xi| \leq R_\rho\}$.

Proof: Define $0 \leq \bar{\delta} < \infty$ and let $x^\delta(\cdot)$ be the trajectory with initial state $x^\delta(0) = x$ corresponding to any δ-optimal disturbance $v^\delta \in \mathcal{L}_2[0,T]$ for the finite horizon value function $V(x,T)$ (2.33), where $\delta \leq \bar{\delta}$, $|x| \leq \rho$, $\rho > \bar{\rho}$, and $\bar{\rho}$ is given by (2.65). Then, by Theorem 2.5.20, $|x^\delta(T)| \leq R_\rho$, where R_ρ is given by (2.67). Hence, as $|x^\delta(T)| \leq R_\rho$,

$$\sup_{\xi \in B_\rho} \left\{V(x,\xi,T)\right\} \ \geq \ V(x,x^\delta(T),T)..$$

But, recalling (2.80) from the proof of Theorem 2.5.24,

$$V^f(x,x^\delta(T),T) \ \geq \ V(x,T) - \delta. \tag{2.82}$$

That is,

$$\sup_{\xi \in B_\rho} \left\{V^f(x,\xi,T)\right\} \ \geq \ V(x,T) - \delta..$$

But, clearly $\sup_{\xi \in \mathbf{R}^n} \left\{V^f(x,\xi,T)\right\} \geq \sup_{\xi \in B_\rho} \left\{V^f(x,\xi,T)\right\}$. Hence, applying (2.77),

$$V(x,T) \ \geq \ \sup_{\xi \in B_\rho} \left\{V^f(x,\xi,T)\right\} \ \geq \ V(x,T) - \delta, \tag{2.83}$$

for any $\delta \in (0, \bar{\delta}]$, completing the proof. ∎

Theorem 2.5.25 is satisfying in that it provides a uniformity property for the finite horizon value function $V(x,T)$ (2.33). That is, for any initial state in a compact set and any time horizon, the finite horizon value function can be obtained by taking the supremum over all possible final states in a defined compact set of the finite horizon fixed final state value function $V^f(x,\xi,T)$ (2.72). This will allow consideration of time horizons tending in the limit to ∞ in Section 2.13.

2.6 The Available Power of Nonlinear Systems

Throughout Section 2.5, the finite horizon value function $V(x,T)$ (2.33) has been interpreted as the maximum energy retrievable from a system starting from an initial state x on the finite time horizon $[0,T]$. Consistent with this interpretation, a notion of the maximum power generated by a system on an *infinite* time horizon follows by simply taking the infinite horizon average of $V(x,T)$ with respect to T. Consequently, the power generated by a system is maximal if perturbed by the optimal disturbance in $V(x,T)$ (2.33) for $T \to \infty$. For suboptimal disturbances, clearly the power generation need not be maximal. Hence, the maximum power generation of a system is referred to as the *available power* of that system.

Definition 2.6.1 *Define the available power λ_a^γ as the maximum power generated by system Σ with initial state $x \in \mathbf{R}^n$ on an infinite time horizon. That is,*

$$\lambda_a^\gamma(x) \;=\; \limsup_{T\to\infty} \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \frac{1}{T}\int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right]\, ds : x(0) = x \right\}. \quad (2.84)$$

Equivalently,

$$\lambda_a^\gamma(x) \;=\; \limsup_{T\to\infty} \left\{ \frac{V_\gamma(x,T)}{T} \right\}, \quad (2.85)$$

where $V_\gamma(x,T)$ is the finite horizon value function $V(x,T)$ (2.33) for gain γ.

A nonlinear system with energy ($\mathcal{L}_2$-) gain does not have the potential to generate power for gains exceeding the $\mathcal{H}_\infty$ norm of that system. This follows since the maximum energy retrievable from a system with energy gain on any horizon is fixed, finite, and independent of the time horizon. This fact is expressed in the following proposition.

Proposition 2.6.2 *Suppose that system Σ has $\mathcal{L}_2$-gain $\le \gamma$. Then, the available power λ_a^γ (2.84) for gain γ is zero. That is, $\lambda_a^\gamma = 0$ for all gains $\gamma \ge \|\Sigma\|_{\mathcal{H}_\infty}$.*

Proof: Given that Σ has $\mathcal{L}_2$-gain $\le \gamma$, Definition 2.20 states that there exists finite nonnegative function $\beta(x)$ such that

$$\int_0^T \left[|z(s)|^2 - \gamma^2|v(s)|^2\right]\, ds \;\le\; \beta(x),$$

for all $v \in \mathcal{L}_2[0,T]$ and all $T \geq 0$. So, taking the supremum over $v \in \mathcal{L}_2[0,T]$ of both sides, and applying the definition of the finite horizon value function $V(x,T)$ (2.33),

$$V(x,T) \leq \beta(x)$$

for all $T \geq 0$. Dividing by $T > 0$ and letting $T \to \infty$, the finiteness of $\beta(x)$ implies that

$$\limsup_{T\to\infty}\left\{\frac{V(x,T)}{T}\right\} \leq \limsup_{T\to\infty}\left\{\frac{\beta(x)}{T}\right\} = 0.$$

But the LHS is, by (2.85), the available power λ_a. Furthermore, the $\mathcal{H}_\infty$-norm $\|\Sigma\|_{\mathcal{H}_\infty}$ is the $\mathcal{L}_2$-gain of Σ. Hence, $\gamma \geq \|\Sigma\|_{\mathcal{H}_\infty}$ implies that Σ has $\mathcal{L}_2$-gain $\leq \gamma$. By the above argument, this implies that $\lambda_a^\gamma = 0$. ∎

Standard observability assumptions in $\mathcal{L}_2$-gain analysis [21, 38] imply that systems with $\mathcal{L}_2$-gain are asymptotically stable in the absence of disturbances. However, many nonlinear systems (including limit cycle systems, see Chapter 5) exhibit oscillatory behaviour in the absence of disturbances. The following remark demonstrates that such systems can have nonzero available power.

Remark 2.6.3 The zero disturbance is suboptimal in the definition of the available power (2.84). Hence, applying the definition of the $\mathcal{FP}$-seminorm (2.11),

$$\begin{aligned}\lambda_a(x) &= \limsup_{T\to\infty}\sup_{v\in\mathcal{L}_2[0,T]}\left\{\frac{1}{T}\int_0^T\left[|z(s)|^2-\gamma^2|v(s)|^2\right]\,ds\right\}\\ &\geq \limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T|z(s)|^2\,ds\right\}\\ &= \|z\|_{\mathcal{FP}}^2.\end{aligned} \tag{2.86}$$

That is, the available power is bounded below by the squared power of the disturbance free output. Consequently, any system with a persistent output of nonzero power has nonzero available power. Note that (2.86) holds with equality if the system is autonomous ($b(x) \equiv 0$). ◄

An important result linking the concepts of power gain property, described by the inequality (2.21), and the available power of a system (2.84) is that any system which exhibits power gain must have finite available power. Furthermore, the available power of a system with power gain $\leq \gamma$ is a lower bound for any power bias for that system.

Theorem 2.6.4 *Suppose that system Σ has $\mathcal{FP}$-gain $\leq \gamma$ with power bias / energy bias pair $(\lambda(x), \beta(x))$. Then, $\lambda_a(x) \leq \lambda(x)$. That is, the available power is the minimal power bias.*

Proof: From the $\mathcal{FP}$-gain inequality (2.34),

$$\frac{V(x,T)}{T} \leq \lambda(x) + \frac{\beta(x)}{T}$$

for all $T > 0$. Taking the limsup as $T \to \infty$ and noting that $\beta(x)$ is finite for every $x \in \mathbf{R}^n$, (2.85) yields that

$$\begin{aligned}
\lambda_a(x) &= \limsup_{T\to\infty} \left\{ \frac{V(x,T)}{T} \right\} \\
&\leq \lambda(x) + \limsup_{T\to\infty} \left\{ \frac{\beta(x)}{T} \right\} \\
&= \lambda(x),
\end{aligned}$$

completing the proof. ∎

Complete reachability of the state space of a system, as defined by Definition 2.2.1, implies that any two states in the state space are reachable from one another by the application of a finite energy disturbance over a finite time interval. However, the power generated by the application of such a finite horizon disturbance prior to the application of an infinite horizon disturbance must be identical to the power generated by the application of the infinite horizon disturbance alone. This follows since any contribution to the energy generated by the system due to the finite horizon disturbance must be lost in the averaging process of computing the power. Hence, it is reasonable to expect that complete reachability implies invariance of the available power with respect to initial conditions, as stated in the following theorem.

Theorem 2.6.5 *Suppose that system Σ has $\mathcal{FP}$-gain $\leq \gamma$. Suppose also that a subset X of the state space of Σ is completely reachable. Then, the available power, $\lambda_a(x)$ (2.84), is invariant with respect to the initial state x for any $x \in X$. That is, $\lambda_a(x) \equiv \lambda_a$.*

Proof: Given $X \subseteq \mathbf{R}^n$ and any $x', x'' \in X$, the aim is to prove that $\lambda_a(x') = \lambda_a(x'')$. Using the complete reachability property, we will show that the system can be controlled from x' to x'' prior to the application of the worst case disturbance, without affecting the available power.

In the definition of complete reachability, let the final time be $\overline{T} < \infty$ and the control be $\overline{v} \in \mathcal{L}_2[-\overline{T}, 0]$ such that $\varphi(0, -\overline{T}, x'; \overline{v}) = x''$. Since Σ has $\mathcal{FP}$-gain $\leq \gamma$, $\overline{v} \in \mathcal{L}_2[-\overline{T}, 0]$ implies that $\overline{z} \in \mathcal{L}_2[-\overline{T}, 0]$, where $\overline{z}$ is the output corresponding to disturbance $\overline{v}$. Hence, immediately we have that

$$\delta := \liminf_{T\to\infty}\left\{\frac{1}{T+\overline{T}}\int_{-\overline{T}}^{0}\left[|\overline{z}(s)|^2-\gamma^2|\overline{v}(s)|^2\right]\,ds\right\} \;=\; 0$$

With $\overline{T}<\infty$, $\limsup a + \liminf b \le \limsup(a+b)$ implies that the available power at state x'' can be written as

$$\begin{aligned}
\lambda_a(x'') &= \limsup_{T\to\infty}\left\{\frac{1}{T+\overline{T}}\sup_{v\in\mathcal{L}_2[0,T]}\left\{\int_0^T\left[|z(s)|^2-\gamma^2|v(s)|^2\right]\,ds : x(0)=x''\right\}\right\}+\delta \\
&\le \limsup_{T\to\infty}\left\{\frac{1}{T+\overline{T}}\sup_{v\in\mathcal{L}_2[0,T]}\left\{\int_0^T\left[|z(s)|^2-\gamma^2|v(s)|^2\right]\,ds : x(0)=x''\right\}+\right. \\
&\qquad\qquad \left.\frac{1}{T+\overline{T}}\int_{-\overline{T}}^{0}\left[|\overline{z}(s)|^2-\gamma^2|\overline{v}(s)|^2\right]\,ds\right\} \\
&\le \limsup_{T\to\infty}\left\{\frac{1}{T+\overline{T}}\sup_{v\in\mathcal{L}_2[0,T]}\left\{\int_0^T\left[|z(s)|^2-\gamma^2|v(s)|^2\right]\,ds : x(0)=x''\right\}+\right. \\
&\qquad \left.\frac{1}{T+\overline{T}}\sup_{\overline{v}\in\mathcal{L}_2[-\overline{T},0]}\left\{\int_{-\overline{T}}^{0}\left[|\overline{z}(s)|^2-\gamma^2|\overline{v}(s)|^2\right]\,ds : \overline{x}(-\overline{T})=x',\overline{x}(0)=x''\right\}\right\} \\
&= \limsup_{T\to\infty}\left\{\frac{1}{T+\overline{T}}\sup_{\tilde{v}\in\mathcal{L}_2[-\overline{T},T]}\int_{-\overline{T}}^{T}\left[|\tilde{z}(s)|^2-\gamma^2|\tilde{v}(s)|^2\right]\,ds : \tilde{x}(-\overline{T})=x'\right\},
\end{aligned} \tag{2.87}$$

where $\tilde{z}\in\mathcal{L}_2[-\overline{T},T]$ is the output corresponding to the augmented disturbance $\tilde{v}\in\mathcal{L}_2[-\overline{T},T]$. But, the RHS of (2.87) is just $\lambda_a(x'')$. That is,

$$\lambda_a(x'') \;=\; \lambda_a(x').$$

As x' and x'' are arbitrary, interchanging x' and x'' yields that $\lambda_a(x')\le\lambda_a(x'')$. Hence, $\lambda_a(x')=\lambda_a(x'')$. ∎

With a stronger form of reachability defined as uniform complete reachabilty (Definition 2.2.2), it is possible using a superadditivity argument and the stability result of Theorem 2.5.20 to prove existence of the limit in the definition of the available power λ_a (Definition 2.6.1). Note that a similar result to Theorem 2.6.6 is proved in [16], but with the use of a stronger (global exponential) stability assumption.

Theorem 2.6.6 *Suppose that assumptions (A3), (A7), (A10), and (A12) hold. Then,*

$$\lambda_a \;=\; \lim_{T\to\infty}\left\{\frac{V(x,T)}{T}\right\} \tag{2.88}$$

uniformly on compact sets.

Proof: This proof utilizes a near-superadditivity argument to demonstrate existence of a limit. To illustrate this argument, consider a nonnegative function $a : \mathbf{R} \to \mathbf{R}$, $a(0) = 0$, where

$$\limsup_{T\to\infty}\left\{\frac{a(T)}{T}\right\} \;=:\; \lambda \;<\; \infty. \tag{2.89}$$

Additionally, suppose that the function $a(\cdot)$ satisfies the near-superadditive property

$$a(s+t) \geq a(s) + a(t) - L, \tag{2.90}$$

for all s, $t \geq 0$, with L a positive constant. Then, by definition of λ (2.89), given any $\varepsilon > 0$, there exists an s such that

$$\frac{a(s)}{s} \;>\; \lambda - \varepsilon, \tag{2.91}$$

$$\frac{L}{s} \;<\; \varepsilon. \tag{2.92}$$

From the near-superadditive property (2.90),

$$\begin{aligned} a(ns) &= a((n-1)s+s) \\ &\geq a((n-1)s) + a(s) - L \\ &\;\vdots \\ &\geq a(0) + na(s) - nL \\ &= na(s) - nL, \end{aligned} \tag{2.93}$$

since $a(0) = 0$. Hence,

$$\frac{a(ns)}{ns} \;\geq\; \frac{a(s)}{s} - \frac{L}{s}.$$

Next, choose any $t > 0$, and set $n := \lfloor \frac{t}{s} \rfloor$, $r = t - ns$. Then, applying (2.90) and (2.93),

$$\begin{aligned} a(t) &= a(ns+r) \\ &\geq a(ns) + a(r) - L \\ &\geq na(s) + a(r) - (n+1)L \\ &\geq na(s) - (n+1)L, \end{aligned}$$

since $a(\cdot)$ is a nonnegative function. Dividing by $t > 0$, and noting that $t \in [ns, (n{+}1)s)$,

$$\begin{aligned} \frac{a(t)}{t} &\geq \frac{na(s)}{t} - \frac{(n+1)L}{t} \\ &\geq \frac{na(s)}{(n+1)s} - \frac{(n+1)L}{ns}. \end{aligned}$$

But, from (2.91), (2.92), this implies that

$$\frac{a(t)}{t} > \left(\frac{n}{n+1}\right)(\lambda-\varepsilon) - \left(\frac{n+1}{n}\right)\varepsilon.$$

Letting $t \to \infty$,

$$\liminf_{t\to\infty}\left\{\frac{a(t)}{t}\right\} \geq \lambda - 2\varepsilon,$$

which holds for any $\varepsilon > 0$. That is, $\liminf_{t\to\infty}\left\{\frac{a(t)}{t}\right\} \geq \lambda$. Hence, combining with (2.89) proves that $\lambda = \lim_{t\to\infty}\{\frac{a_t}{t}\}$.

By demonstrating that the finite horizon value function $V(x,T)$ (2.33) satisfies a super-additive relation similar to (2.93), it is possible to apply the above argument to prove (2.88) given the stated assumptions of the theorem.

Define any $\varepsilon > 0$, $\bar{\delta} > 0$, and $\delta \in (0,\bar{\delta})$. Applying assumptions (A7), (A10), and (A12), Theorem 2.5.20 states that there exists a $\bar{\rho}$ (2.65) such that with $\rho > \bar{\rho}$ and $x_0 \in B_\rho$ (2.63), the final state $x^\delta(S)$ of the trajectory $x^\delta(\cdot)$ corresponding to a δ-optimal disturbance for $V(x_0,S)$ is confined to a compact set defined by ρ. That is,

$$x_0 \in B_\rho \quad \Rightarrow \quad x^\delta(S) \in B_{R_\rho},$$

where B_{R_ρ} (2.62) is the ball of radius R_ρ (2.67). Furthermore, since B_{R_ρ} is compact, the uniform complete reachability assumption (A3) implies that given $x_0 \in B_\rho \subseteq B_{R_\rho}$, there exists T_ρ, $E_\rho < \infty$ such that for any $y \in B_{R_\rho}$, x_0 is reachable from y in time $T_y \leq T_\rho$ by the application of a finite energy disturbance $v_y \in \mathcal{L}_2[0,T_y]$, where $\|v_y\|_{\mathcal{L}_2[0.T_y]} \leq E_\rho$. Hence,

$$\begin{aligned}\inf_{y\in B_{R_\rho}}\left\{\int_0^{T_y}\left[c(x_y(s)) - \gamma^2|v_y(s)|^2\right]\,ds\right\} &\geq -\gamma^2\sup_{y\in B_{R_\rho}}\left\{\int_0^{T_y}|v_y(s)|^2\,ds\right\}\\ &=: -L_\rho,\end{aligned}$$

and $L_\rho \leq \gamma^2 E_\rho < \infty$.

Since $\lambda_a = \limsup_{T\to\infty}\left\{\frac{V(x,T)}{T}\right\}$, define a time horizon $S < \infty$ sufficiently large such that

$$\frac{\bar{\delta}}{S} < \varepsilon, \tag{2.94}$$

$$\frac{T_\rho}{S} < \varepsilon, \tag{2.95}$$

$$\frac{L_\rho}{S} < \varepsilon, \text{ and} \tag{2.96}$$

$$\frac{V(x_0,S)}{S} > \lambda_a - \varepsilon. \tag{2.97}$$

With $v^\delta(\cdot)$ a δ-optimal disturbance for $V(x_0, S)$ with gain γ on the interval $[0, S]$ with $x^\delta(0) = x_0$ (where $x^\delta(\cdot)$ is trajectory corresponding to $v^\delta(\cdot)$), fix $y := x^\delta(S)$ and define the switching disturbance

$$\bar{v}^\delta(t) = \begin{cases} v^\delta(t) & t \in [0, S) \\ v_y(t - S) & t \in [S, T) \end{cases} \tag{2.98}$$

where $T := S + T_y$. Given that $\bar{x}^\delta(0) = x_0$, the definition of the disturbance $v_y \in \mathcal{L}_2[0, T_y]$ implies that $\bar{x}^\delta(T) = x_0$. Next, repeat $\bar{v}^\delta(\cdot)$, yielding the periodic switching disturbance $\tilde{v}^\delta(\cdot)$. That is, set

$$\tilde{v}^\delta(t) = \bar{v}^\delta(t - nT)$$

where $n := \lfloor \frac{t}{T} \rfloor$ (see Figure 2.5). Fix $t < \infty$, and define $r = t - nT$.

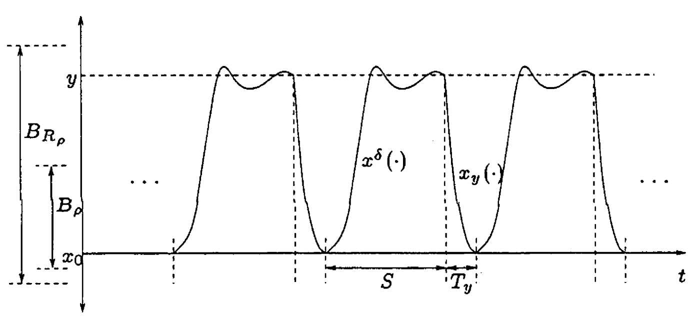


Figure 2.5: Periodic trajectory $\tilde{x}^\delta(\cdot)$ due to the switching disturbance $\tilde{v}^\delta(\cdot)$

Since $\tilde{v}^\delta(\cdot)$ is suboptimal on any time horizon,

$$\begin{aligned} V(x_0, t) &\geq \int_0^t \left[c(\tilde{x}^\delta(s)) - \gamma^2 |\tilde{v}^\delta(s)|^2 \right] ds \\ &= \int_0^{nT} \left[c(\tilde{x}^\delta(s)) - \gamma^2 |\tilde{v}^\delta(s)|^2 \right] ds + \int_0^r \left[c(\tilde{x}^\delta(s)) - \gamma^2 |\tilde{v}^\delta(s)|^2 \right] ds. \end{aligned} \tag{2.99}$$

Furthermore, $\tilde{x}^\delta(kT) = x_0$ for any integer $k \geq 0$, so that

$$\begin{aligned} &\int_0^{nT} \left[c(\tilde{x}^\delta(s)) - \gamma^2 |\tilde{v}^\delta(s)|^2 \right] ds \\ &= \sum_{k=0}^{n-1} \int_{kT}^{(k+1)T} \left[c(\tilde{x}^\delta(s)) - \gamma^2 |\tilde{v}^\delta(s)|^2 \right] ds \\ &= \sum_{k=0}^{n-1} \int_{kT}^{kT+S} \left[c(x^\delta(s)) - \gamma^2 |v^\delta(s)|^2 \right] ds + \sum_{k=0}^{n-1} \int_{kT+S}^{(k+1)T} \left[c(x_y(s)) - \gamma^2 |v_y(s)|^2 \right] ds \\ &= n \int_0^S \left[c(x^\delta(s)) - \gamma^2 |v^\delta(s)|^2 \right] ds + n \int_S^T \left[c(x_y(s)) - \gamma^2 |v_y(s)|^2 \right] ds. \end{aligned} \tag{2.100}$$

But, $v^\delta(\cdot)$ is δ-optimal for gain γ, initial state x_0, and interval $[0, S]$. Hence,

$$\int_0^S \left[c(x^\delta(s)) - \gamma^2|v^\delta(s)|^2\right] ds \;\geq\; V(x_0, S) - \delta. \tag{2.101}$$

Furthermore, from (2.94),

$$\int_0^{T_y} \left[c(x_y(s)) - \gamma^2|v_y(s)|^2\right] ds \;\geq\; -L_\rho. \tag{2.102}$$

So, using (2.101), (2.102) in (2.100),

$$\int_0^{nT} \left[c(\tilde{x}^\delta(s)) - \gamma^2|\tilde{v}^\delta(s)|^2\right] ds \;\geq\; nV(x_0, S) - n(L_\rho + \delta). \tag{2.103}$$

The remaining integral in (2.99) is

$$\begin{aligned}
&\int_0^r \left[c(\tilde{x}^\delta(s)) - \gamma^2|\tilde{v}^\delta(s)|^2\right] ds \\
&= \int_0^{r\wedge S} \left[c(x^\delta(s)) - \gamma^2|v^\delta(s)|^2\right] ds + \int_{r\wedge S}^r \left[c(x_y(s)) - \gamma^2|v_y(s)|^2\right] ds.
\end{aligned} \tag{2.104}$$

If $r > S$, then this integral term (2.104) is bounded below by $-L_\rho$. Alternatively, if $r \leq S$, the dynamic programming equation (2.40) implies that

$$\begin{aligned}
\int_0^r \left[c(\tilde{x}^\delta(s)) - \gamma^2|\tilde{v}^\delta(s)|^2\right] ds \;&\geq\; V(x_0, S) - V(x^\delta(r), S - r) - \delta \\
&\geq\; -V(x^\delta(r), S - r) - \delta.
\end{aligned} \tag{2.105}$$

But, since $S < \infty$, Lemma 2.5.16 implies that there exists an $R'_\rho < \infty$ such that $x^\delta(r) \in B_{R'_\rho}$. So, Theorem 2.5.6 implies that there exists an $M < \infty$ such that $V(x^\delta(r), S - r) \leq M$ for any $r \in [0, S]$. That is, (2.105) becomes

$$\int_0^r \left[c(\tilde{x}^\delta(s)) - \gamma^2|\tilde{v}^\delta(s)|^2\right] ds \;\geq\; -M - \delta. \tag{2.106}$$

Combining the $r > S$ case and the $r \leq S$ case (2.106) in (2.104) yields that

$$\int_0^r \left[c(\tilde{x}^\delta(s)) - \gamma^2|\tilde{v}^\delta(s)|^2\right] ds \;\geq\; -M - \delta - L_\rho \tag{2.107}$$

for any $r \in [0, T]$. Combining (2.103), (2.107), and (2.99) then yields the near super-additive relation

$$V(x_0, t) \;\geq\; nV(x_0, S) - (n+1)(L_\rho + \delta) - M. \tag{2.108}$$

Dividing both sides by $t > 0$, and noting that $t \in [nT, (n+1)T)$,

$$\begin{aligned}
\frac{V(x_0, t)}{t} \;&\geq\; \frac{nV(x_0, S)}{(n+1)T} - \frac{(n+1)(L_\rho + \delta)}{nT} - \frac{M}{nT} \\
&=\; \frac{nV(x_0, S)}{(n+1)S}\frac{S}{T} - \frac{(n+1)(L_\rho + \delta)}{nT} - \frac{M}{nT} \\
&=\; \frac{nV(x_0, S)}{(n+1)S}\left(1 - \frac{T_y}{T}\right) - \frac{(n+1)(L_\rho + \delta)}{nT} - \frac{M}{nT}
\end{aligned}$$

$$\geq \frac{nV(x_0,S)}{(n+1)S}\left(1-\frac{T_\rho}{S}\right)-\frac{(n+1)(L_\rho+\bar{\delta})}{nS}-\frac{M}{nS} \tag{2.109}$$

Now, applying (2.94) through (2.97), inequality (2.109) becomes

$$\frac{V(x_0,t)}{t} > \left(\frac{n}{n+1}\right)(\lambda_a-\varepsilon)(1-\varepsilon)-\left(\frac{n+1}{n}\right)2\varepsilon-\frac{M}{nS},$$

for any $x_0 \in B_\rho$. Sending $n \uparrow \infty$ and recalling that $M < \infty$ yields

$$\liminf_{t\to\infty}\left\{\frac{V(x_0,t)}{t}\right\} > (\lambda_a-\varepsilon)(1-\varepsilon)-2\varepsilon$$

for arbitrarily small $\varepsilon > 0$. Hence, (2.88) holds for any initial state $x_0 \in B_\rho$, for any $\rho > \bar{\rho}$ arbitrarily large, completing the proof. ∎

With existence of the limit in the definition of the available power λ_a (2.84), differentiability of the finite horizon value function $V(x,T)$ (2.33) implies an alternative form for the available power, as expressed in the following result.

Theorem 2.6.7 *Suppose that assumptions (A3), (A7), (A10), and (A12) hold. Suppose additionally that the finite horizon value function $V(x,T)$ (2.33) is differentiable with respect to $T > 0$, and that the limit*

$$\Lambda(x) = \lim_{T\to\infty}\left\{\frac{\partial V}{\partial T}(x,T)\right\} \tag{2.110}$$

exists and is finite. Then, $\Lambda(x) \equiv \lambda_a(x)$.

Proof: Suppose that the limit (2.110) exists and is finite. Then, given $\varepsilon > 0$, there exists a $T^* > 0$ such that

$$\begin{aligned} T > T^* \quad &\Rightarrow \quad \left|\frac{\partial V}{\partial T}(x,T)-\Lambda(x)\right| < \varepsilon \\ &\Leftrightarrow \quad \Lambda(x)-\varepsilon < \frac{\partial V}{\partial T}(x,T) < \Lambda(x)+\varepsilon \end{aligned} \tag{2.111}$$

Let $T > T_0 > T^*$. By the Mean Value Theorem, there exists a $\tau \in (T_0,T)$ such that

$$\frac{\partial V}{\partial T}(x,\tau)\,[T-T_0] = V(x,T)-V(x,T_0) \tag{2.112}$$

As the first of two cases, suppose that $V(x,T) \not\to \infty$ as $T \to \infty$. Since $V(x,T)$ is nondecreasing in T by Proposition 2.5.2, this implies that $V(x,T)$ is bounded above by some $K \geq 0$ for all $T \geq 0$, such that $\lim_{T\to\infty}\{V(x,T)\} = K$. Now, rewriting (2.112),

$$\frac{\partial V}{\partial T}(x,\tau) = \frac{V(x,T)}{T}\underbrace{\left(\frac{T}{T-T_0}\right)}_{\mu_1(T)}-\underbrace{\frac{V(x,T_0)}{T-T_0}}_{\mu_2(x,T_0,T)}. \tag{2.113}$$

By inspection, $\lim_{T\to\infty}\{\mu_1(T)\} = 1$. Furthermore, since $V(x,T) \leq K$ for all $T \geq 0$, $\lim_{T\to\infty}\{\mu_2(x,T_0,T)\} = 0$. So, there exists a T^{**} such that

$$\begin{aligned} T > T^{**} \quad &\Rightarrow \quad |\mu_1(T) - 1| < \varepsilon \text{ and } |\mu_2(x,T_0,T)| < \varepsilon \\ &\Leftrightarrow \quad 1-\varepsilon < \mu_1(T) < 1+\varepsilon \text{ and } -\varepsilon < \mu_2(x,T_0,T) < \varepsilon \end{aligned} \tag{2.114}$$

Let $\varepsilon \in (0,1)$, and choose $T > T_0 > \max(T^*, T^{**})$ in (2.112), (2.113). Then, combining (2.113) and (2.114),

$$\frac{V(x,T)}{T}(1-\varepsilon) - \varepsilon \; < \; \frac{\partial V}{\partial T}(x,\tau) \; < \; \frac{V(x,T)}{T}(1+\varepsilon) + \varepsilon.$$

Rearranging,

$$\frac{1}{1+\varepsilon}\left(\frac{\partial V}{\partial T}(x,\tau) - \varepsilon\right) \; < \; \frac{V(x,T)}{T} \; < \; \frac{1}{1-\varepsilon}\left(\frac{\partial V}{\partial T}(x,\tau) + \varepsilon\right). \tag{2.115}$$

Since $T > \tau > T_0 > \max(T^*, T^{**})$, (2.111) can be applied for the horizon τ. Hence, combining (2.111) and (2.115),

$$\begin{aligned} \frac{\Lambda(x) - 2\varepsilon}{1+\varepsilon} \; &< \; \frac{V(x,T)}{T} \; < \; \frac{\Lambda(x) + 2\varepsilon}{1-\varepsilon} \\ \Rightarrow \; \Lambda(x) - \varepsilon\left(\frac{\Lambda(x)+2}{1+\varepsilon}\right) \; &< \; \frac{V(x,T)}{T} \; < \; \Lambda(x) + \varepsilon\left(\frac{\Lambda(x)+2}{1-\varepsilon}\right) \\ \Rightarrow \; \Lambda(x) - \varepsilon\left(\frac{\Lambda(x)+2}{1-\varepsilon}\right) \; &< \; \frac{V(x,T)}{T} \; < \; \Lambda(x) + \varepsilon\left(\frac{\Lambda(x)+2}{1-\varepsilon}\right). \end{aligned}$$

Hence, given $\varepsilon \in (0,1)$, there exists a $\bar{T} = \max(T^*, T^{**})$ such that

$$T > \bar{T} \;\Rightarrow\; \left|\frac{V(x,T)}{T} - \Lambda(x)\right| \; < \; \varepsilon\left(\frac{\Lambda(x)+2}{1-\varepsilon}\right).$$

Since $\Lambda(x)$ is finite for each x, this implies that $\lambda_a = \lim_{T\to\infty}\left\{\frac{V(x,T)}{T}\right\} = \Lambda(x)$. Note that since $V(x,T) \leq K$ for all $T \geq 0$, this implies that $\lambda_a = 0 = \Lambda(x)$.

Now for the second case. Suppose that that $V(x,T) \to \infty$ as $T \to \infty$. Rewriting (2.112), there exists a $\tau \in (T_0, T)$, $T > T_0$ such that

$$\frac{\partial V}{\partial T}(x,\tau) \; = \; \frac{V(x,T)}{T}\underbrace{\left(\frac{1 - \frac{V(x,T_0)}{V(x,T)}}{1 - \frac{T_0}{T}}\right)}_{\mu(x,T_0,T)} \tag{2.116}$$

Since $V(x,T) \to \infty$ as $T \to \infty$, by inspection, $\lim_{T\to\infty}\{\mu(x,T_0,T)\} = 1$, where T_0 and x are fixed. So, there exists a $T^{**} > 0$ such that

$$\begin{aligned} T > T^{**} \quad &\Rightarrow \quad |\mu(x,T_0,T) - 1| < \varepsilon \\ &\Leftrightarrow \quad 1 - \varepsilon < \mu(x,T_0,T) < 1+\varepsilon \end{aligned} \tag{2.117}$$

Let $\varepsilon \in (0,1)$ and choose $T > T_0 > \max(T^*, T^{**})$ in (2.111), (2.117). Then, combining (2.116) and (2.117),

$$\frac{V(x,T)}{T}(1-\varepsilon) \quad < \quad \frac{\partial V}{\partial T}(x,\tau) \quad < \quad \frac{V(x,T)}{T}(1+\varepsilon).$$

Rearranging,

$$\frac{\partial V}{\partial T}(x,\tau)\left(\frac{1}{1+\varepsilon}\right) < \frac{V(x,T)}{T} < \frac{\partial V}{\partial T}(x,\tau)\left(\frac{1}{1-\varepsilon}\right). \tag{2.118}$$

Since $T > \tau > T_0 > \max(T^*, T^{**})$, (2.111) can again be applied for horizon τ. Hence, combining (2.111) and (2.118),

$$\begin{aligned} \frac{\Lambda(x)-\varepsilon}{1+\varepsilon} \quad &< \quad \frac{V(x,T)}{T} \quad < \quad \frac{\Lambda(x)+\varepsilon}{1-\varepsilon} \\ \Rightarrow \Lambda(x) - \varepsilon\left(\frac{\Lambda(x)+1}{1-\varepsilon}\right) \quad &< \quad \frac{V(x,T)}{T} \quad < \quad \Lambda(x) + \varepsilon\left(\frac{\Lambda(x)+1}{1-\varepsilon}\right). \end{aligned}$$

Hence, given $\varepsilon \in (0,1)$, there exists a $\bar{T} = \max(T^*, T^{**})$ such that

$$T > \bar{T} \quad \Rightarrow \quad \left|\frac{V(x,T)}{T} - \Lambda(x)\right| < \varepsilon\left(\frac{\Lambda(x)+1}{1-\varepsilon}\right).$$

Since $\Lambda(x)$ is finite for each x, this implies that the limit $\lim_{T\to\infty}\left\{\frac{V(x,T)}{T}\right\} = \Lambda(x)$ exists and is finite. But, by definition (2.84), this limit is also the available power $\lambda_a(x)$. ■

Theorems 2.6.6 and 2.6.7 are particularly useful when endeavouring to compute numerical approximations for the available power λ_a (2.84). Without the existence of the limit in (2.84), there is no guarantee that a sequence of approximations of the ratio $\frac{V(x,T)}{T}$ tends as $T \to \infty$ to the limsup. Furthermore, the alternative form for λ_a given by Theorem 2.6.7 provides the key to improving the rate of convergence of available power approximations. These issues are discussed in greater detail in Chapter 4.

One of the problems with Definition 2.6.1 of the available power is that (2.84) does not engender a simple notion of the "worst case" disturbance which excites the maximum power generation of the system. Rather, (2.84) defines a sequence of finite horizon worst case disturbances which in the limit of the infinite horizon give rise to the maximum power generation of the system. Hence, for the remainder of this section we focus on defining a more satisfactory infinite horizon worst case disturbance. Utimately, this involves finding an alternative expression for the available power which is the result of a supremum over a class of infinite horizon disturbances. This alternative form for the available power is provided by the following theorem (similar to Theorem 5.4 of [16]).

Theorem 2.6.8 *Suppose that assumptions (A3), (A7), (A10), and (A12) hold. Then,*

$$\lambda_a = \sup_{v\in\mathcal{FP}} \lim_{T\to\infty} \left\{ \frac{1}{T}\int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds : x(0) = x \right\}, \tag{2.119}$$

where $\mathcal{FP}$ is the space of finite power signals (2.12).

Proof: Applying Proposition 2.3.7 to the definition of available power λ_a (2.84),

$$\begin{aligned}
\lambda_a &= \limsup_{T\to\infty} \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \frac{1}{T}\int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds : x(0) = x \right\} \\
&\geq \limsup_{T\to\infty} \sup_{v\in\mathcal{FP}} \left\{ \frac{1}{T}\int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds : x(0) = x \right\} \\
&\geq \sup_{v\in\mathcal{FP}} \limsup_{T\to\infty} \left\{ \frac{1}{T}\int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds : x(0) = x \right\}, \qquad (2.120)
\end{aligned}$$

which proves inequality in (2.119) in one direction.

To prove the opposite direction, we follow the proof of Theorem 2.6.6. Recalling (2.103),

$$\int_0^{nT} \left[c(\tilde{x}^\delta(s)) - \gamma^2|\tilde{v}^\delta(s)|^2\right] ds \;\geq\; nV(x_0, S) - n(L_\rho + \delta), \tag{2.121}$$

where $\bar{v}^\delta \in \mathcal{L}_2[0, nT]$ is the switching disturbance (2.98). Hence,

$$\begin{aligned}
\frac{1}{nT}\int_0^{nT} \left[c(\tilde{x}^\delta(s)) - \gamma^2|\tilde{v}^\delta(s)|^2\right] ds &\geq \frac{V(x_0,S)}{S}\frac{S}{T} - \frac{L_\rho}{T} - \frac{\bar{\delta}}{T} \\
&= \frac{V(x_0,S)}{S}\left(1 - \frac{T_y}{T}\right) - \frac{L_\rho}{T} - \frac{\bar{\delta}}{T} \\
&\geq \frac{V(x_0,S)}{S}\left(1 - \frac{T_\rho}{S}\right) - \frac{L_\rho}{S} - \frac{\bar{\delta}}{S}. \qquad (2.122)
\end{aligned}$$

Applying (2.94) through (2.97), (2.122) implies that for any $v \in \mathcal{FP}$,

$$\begin{aligned}
&\liminf_{t\to\infty} \left\{ \frac{1}{t}\int_0^t \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds \right\} \\
&\qquad \geq \lim_{n\to\infty} \left\{ \frac{1}{nT}\int_0^{nT} \left[c(\tilde{x}^\delta(s)) - \gamma^2|\tilde{v}^\delta(s)|^2\right] ds \right\} \\
&\qquad \geq (\lambda_a - \varepsilon)(1-\varepsilon) - 2\varepsilon,
\end{aligned}$$

for any $\varepsilon > 0$. Noting that $x(0) = x$, this implies that

$$\sup_{v\in\mathcal{FP}} \liminf_{t\to\infty} \left\{ \frac{1}{t}\int_0^t \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds : x(0) = x \right\} \;\geq\; \lambda_a. \tag{2.123}$$

Combining inequalities (2.120) and (2.123) completes the proof. ∎

Theorem 2.6.8 then facilitates the definition of an infinite horizon worst case disturbance.

Definition 2.6.9 *Define the cost*

$$J(x,v) \;=\; \limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2\right]\, ds : x(0)=x\right\}. \tag{2.124}$$

Then, the infinite horizon worst case $\mathcal{FP}$ disturbance is defined as $v^* \in \mathcal{FP}$, where

$$\lambda_a \;=\; \sup_{v\in\mathcal{FP}}\{J(x,v)\} \;=\; J(x,v^*). \tag{2.125}$$

Using this definition, the power of the worst case disturbance and corresponding output can be computed in terms of the derivative (if it exists) of the available power λ_a^γ with respect to the gain γ. For this, we need the following simple lemma.

Lemma 2.6.10 *The available power λ_a^γ (2.84) is a nonincreasing function of the gain γ.*

Proof: Immediate from Proposition 2.5.3 and (2.85). ∎

Theorem 2.6.11 *Suppose that assumptions (A3), (A7), (A10), and (A12) hold. Let $\gamma^* < \hat{\gamma} < \gamma < \tilde{\gamma}$ and suppose that the available power is finite for gain γ^* and differentiable for gain γ. Suppose also that $V_{\gamma^*}(x,T)$ is finite for all $T<\infty$, and that the infinite horizon worst case $\mathcal{FP}$ disturbance $v^*\in\mathcal{FP}$ (2.125) exists. Then, the power of this worst case disturbance and the corresponding output is given by*

$$\|v_\gamma^*\|_{\mathcal{FP}} \;=\; \sqrt{\frac{-1}{2\gamma}\left(\frac{d\lambda_a^\gamma}{d\gamma}\right)} \tag{2.126}$$

$$\|z_\gamma^*\|_{\mathcal{FP}} \;=\; \sqrt{\frac{-\gamma^3}{2}\frac{d}{d\gamma}\left(\frac{\lambda_a^\gamma}{\gamma^2}\right)} \tag{2.127}$$

Proof: By Proposition 2.5.3, finiteness of the finite horizon value function $V(x,T)$ is guaranteed for gains $\hat{\gamma}<\gamma<\tilde{\gamma}$, since $0 \le V_{\tilde{\gamma}}(x,T) \le V_\gamma(x,T) \le V_{\hat{\gamma}}(x,T) \le V_{\gamma^*}(x,T) < \infty$.

Assuming that the worst case $\mathcal{FP}$ disturbance $v_\gamma^*\in\mathcal{FP}$ (2.125) exists for gain γ, Theorem 2.6.8 implies that

$$\lambda_a^\gamma \;=\; \lim_{T\to\infty}\left\{\int_0^T \left[c(x_\gamma^*(s)) - \gamma^2|v_\gamma^*(s)|^2\right]\, ds\right\}. \tag{2.128}$$

However, Theorem 2.6.6 states that

$$\lambda_a^\gamma \;=\; \lim_{T\to\infty}\left\{\frac{V_\gamma(x,T)}{T}\right\}. \tag{2.129}$$

Hence, combining (2.128) and (2.129),

$$\lim_{T\to\infty}\left\{\frac{V_\gamma(x,T)}{T} - \frac{1}{T}\int_0^T \left[c(x_\gamma^*(s)) - \gamma^2|v_\gamma^*(s)|^2\right]\, ds\right\} \;=\; 0.$$

So, given $\delta > 0$, there exists $T^* > 0$ such that

$$T > T^* \quad \Rightarrow \quad V_\gamma(x,T) \leq \int_0^T \left[c(x_\gamma^*(s)) - \gamma^2 |v_\gamma^*(s)|^2\right] \, ds + \delta T. \tag{2.130}$$

But, $v_\gamma^* \in \mathcal{FP}$ is suboptimal on the interval $[0,T]$ for the definition of $V_{\tilde{\gamma}}(x,T)$. That is,

$$\int_0^T \left[c(x_\gamma^*(s)) - \tilde{\gamma}^2 |v_\gamma^*(s)|^2\right] \, ds \;\; \leq \;\; V_{\tilde{\gamma}}(x,T). \tag{2.131}$$

So, combining (2.130) and (2.131),

$$T > T^* \quad \Rightarrow \quad (\gamma^2 - \tilde{\gamma}^2) \int_0^T |v_\gamma^*(s)|^2 \, ds \leq V_{\tilde{\gamma}}(x,T) - V_\gamma(x,T) + \delta T.$$

Dividing by $T > T^*$, letting $T \to \infty$, applying Theorem 2.6.6, and noting that $\gamma < \tilde{\gamma}$ yields that

$$\|v_\gamma^*\|_{\mathcal{FP}}^2 \;\; \geq \;\; \frac{\lambda_a^\gamma - \lambda_a^{\tilde{\gamma}}}{\tilde{\gamma}^2 - \gamma^2} - \frac{\delta}{\tilde{\gamma}^2 - \gamma^2},$$

for any $\delta > 0$ and any $\tilde{\gamma} > \gamma$. Hence,

$$\begin{aligned} \|v_\gamma^*\|_{\mathcal{FP}}^2 \;\; &\geq \;\; \frac{-1}{2\gamma} \lim_{\tilde{\gamma} \downarrow \gamma} \left\{ \frac{\lambda_a^{\tilde{\gamma}} - \lambda_a^\gamma}{\tilde{\gamma} - \gamma} \right\} \\ &= \;\; \frac{-1}{2\gamma} \left(\frac{d\lambda_a^\gamma}{d\gamma} \right), \end{aligned} \tag{2.132}$$

since the available power is differentiable with respect to gain for gain γ. Similarly, $v_\gamma^* \in \mathcal{FP}$ is suboptimal on the interval $[0,T]$ for the definition of $V_{\hat{\gamma}}$. Following the same procedure,

$$\begin{aligned} \|v_\gamma^*\|_{\mathcal{FP}}^2 \;\; &\leq \;\; \frac{-1}{2\gamma} \lim_{\hat{\gamma} \uparrow \gamma} \left\{ \frac{\lambda_a^\gamma - \lambda_a^{\hat{\gamma}}}{\gamma - \hat{\gamma}} \right\} \\ &= \;\; \frac{-1}{2\gamma} \left(\frac{d\lambda_a^\gamma}{d\gamma} \right). \end{aligned} \tag{2.133}$$

Combining inequalities (2.132) and (2.133) yield the required result (2.126). A similar argument yields $\|z_\gamma^*\|_{\mathcal{FP}}^2$ (2.127). Note that Lemma 2.6.10 ensures that the derivatives on the RHS of (2.126) and (2.127) are nonnegative, so that the the $\mathcal{FP}$ seminorms $\|v_\gamma^*\|_{\mathcal{FP}}$ and $\|z_\gamma^*\|_{\mathcal{FP}}$ well defined. ∎

Immediate it is clear from Theorem 2.6.11 that systems with available power independent of the gain γ exhibit infinite horizon worst case $\mathcal{FP}$ disturbances with zero power. Systems with with finite $\mathcal{L}_2$-gain ($\lambda_a = 0$, Proposition 2.6.2) for example exhibit a finite energy worst case disturbance and output.

Corollary 2.6.12 *The available power λ_a^γ is given by*

$$\lambda_a^\gamma = \|z_\gamma^*\|_{\mathcal{FP}}^2 - \gamma^2\|v_\gamma^*\|_{\mathcal{FP}}^2, \tag{2.134}$$

where $\|v_\gamma^*\|_{\mathcal{FP}}$, $\|z_\gamma^*\|_{\mathcal{FP}}$ are given by (2.126), (2.127) respectively.

Proof: Applying Theorem 2.6.11 equations (2.126) and (2.127),

$$\begin{aligned}
\|z_\gamma^*\|_{\mathcal{FP}}^2 - \gamma^2\|v_\gamma^*\|_{\mathcal{FP}}^2 &= \frac{-\gamma^3}{2}\frac{d}{d\gamma}\left(\frac{\lambda_a^\gamma}{\gamma^2}\right) + \frac{\gamma}{2}\left(\frac{d\lambda_a^\gamma}{d\gamma}\right) \\
&= \frac{-\gamma^3}{2}\left(\frac{\gamma^2\frac{d\lambda_a^\gamma}{d\gamma} - 2\gamma\lambda_a^\gamma}{\gamma^4}\right) + \frac{\gamma}{2}\left(\frac{d\lambda_a^\gamma}{d\gamma}\right) \\
&= -\frac{\gamma}{2}\left(\frac{d\lambda_a^\gamma}{d\gamma}\right) + \lambda_a^\gamma + \frac{\gamma}{2}\left(\frac{d\lambda_a^\gamma}{d\gamma}\right) \\
&= \lambda_a^\gamma
\end{aligned}$$

■

Corollary 2.6.12 leads to the following more general result.

Theorem 2.6.13 *Suppose that the available power for gain γ^* is finite and derivative exists for gain $\gamma > \gamma^*$. Then, the available power λ_a^γ is given by*

$$\lambda_a^\gamma = \sup_{v\in\mathcal{FP}} \left\{\|z\|_{\mathcal{FP}}^2 - \gamma^2\|v\|_{\mathcal{FP}}^2\right\}. \tag{2.135}$$

Proof: Since λ_a^γ is differentiable for gain γ and is nonincreasing with respect to gain, $\|v_\gamma^*\|_{\mathcal{FP}}$ as in (2.126) must be finite. Hence, $v_\gamma^* \in \mathcal{FP}$. Thus, applying Corollary 2.6.12,

$$\begin{aligned}
\lambda_a^\gamma &= \|z_\gamma^*\|_{\mathcal{FP}}^2 - \gamma^2\|v_\gamma^*\|_{\mathcal{FP}}^2 \\
&\leq \sup_{v\in\mathcal{FP}} \left\{\|z\|_{\mathcal{FP}}^2 - \gamma^2\|v\|_{\mathcal{FP}}^2\right\} \qquad (2.136)
\end{aligned}$$

But, Proposition 2.3.7 states that $\mathcal{FP} \in \mathcal{L}_{2e}$. So, from the definition of available power (2.84),

$$\begin{aligned}
\lambda_a^\gamma &\geq \limsup_{T\to\infty} \sup_{v\in\mathcal{FP}} \left\{\frac{1}{T}\int_0^T \left[|z(s)|^2 - \gamma^2|v(s)|^2\right] ds\right\} \\
&\geq \sup_{v\in\mathcal{FP}} \limsup_{T\to\infty} \left\{\frac{1}{T}\int_0^T \left[|z(s)|^2 - \gamma^2|v(s)|^2\right] ds\right\} \\
&\geq \sup_{v\in\mathcal{FP}} \left\{\limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T |z(s)|^2\, ds\right\} - \gamma^2 \limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T |v(s)|^2\, ds\right\}\right\} \\
&= \sup_{v\in\mathcal{FP}} \left\{\|z\|_{\mathcal{FP}}^2 - \gamma^2\|v\|_{\mathcal{FP}}^2\right\} \qquad (2.137)
\end{aligned}$$

Combining inequalities (2.136) and (2.137) completes the proof. ■

The available power is inherently a notion which applies to systems with power gain. Indeed, Theorem 2.6.4 states that the available power is a lower bound for all possible power biases for a system with power gain $\leq \gamma$, which by Proposition 2.6.2, collapses to zero in the case of systems with $\mathcal{L}_2$ gain. The applicability of the available power concept to systems with power gain is further highlighted with Theorem 2.6.11. In this theorem, it is made clear that the power of the "worst case" disturbance which excites the maximum power generation of the system depends explicitly on how the available power varies with gain. Clearly, any system with zero available power (including systems with finite $\mathcal{L}_2$-gain) or indeed constant available power (autonomous systems for example) must exhibit a zero power worst case disturbance. Alternatively, systems with gain varying available power, such as affine systems (Section 5.3) and limit cycle systems (Sections 5.7 and 5.8) must exhibit worst case disturbances with nonzero power. It is this nonzero power which will be seen to yield fundamental differences in the dynamics of a power gain system when moving from the absence disturbances to the presence of the worst case disturbance.

2.7 Power Dissipative Nonlinear Systems

The fundamental machinery underlying $\mathcal{L}_2$-gain analysis [38] and nonlinear $\mathcal{H}_\infty$ control is the theory *energy dissipative systems* [42, 21, 22]. One important property of dissipative systems is the ability to store energy, subject to losses (or *dissipation*). When nonempty, the reservoir of stored energy (or *storage*) may be called upon to deliver energy to the external environment via the input and output. However, due to finiteness of the storage, such delivery of energy cannot continue indefinitely in the absence of inputs. That is, energy dissipative systems are limited to delivering a finite amount of energy in the absence of disturbances.

In treating systems with finite power ($\mathcal{FP}$-) gain, it is apparent that such systems may deliver an infinite amount of energy in the absence of disturbances (Remark 2.4.5). Consequently, in general, systems with power gain need not be energy dissipative, and thus may not amenable to the application of conventional energy dissipative systems theory.

Since it is the internal power generation abilities of power gain systems which proscribes the application of energy dissipative systems theory, a simple yet effective approach is to account for this power generation when establishing an energy balance for the system. Since the power bias λ is a measure of the power generation for systems with $\mathcal{FP}$-gain, this accounting can be done by subtracting the expected energy generation λT from the energy *supply rate*. This yields the following generalization of the energy dissipativity.

Definition 2.7.1 *System Σ is power dissipative if there exists a finite power bias / storage function pair (λ, V), where λ and V are nonnegative, satisfying the dissipation inequality*

$$V(x) + \int_0^T \left[\gamma^2 |v(s)|^2 - c(x(s)) + \lambda\right] \, ds \;\; \geq \;\; V(x(T)) \tag{2.138}$$

for all $v \in \mathcal{L}_2[0,T]$ and all $T \geq 0$, where $x(0) = x$.

Hence, systems which are power dissipative are accompanied by a power bias / storage function pair (λ, V) which satisfies the dissipation inequality, rather than just a storage function V as in the case of energy dissipative systems. Furthermore, note that any energy dissipative system is also power dissipative, since the power bias may be chosen arbitrarily.

An important property of the dissipation inequality is that it does not uniquely specify the power bias / storage pair (λ, V). Rather, a whole range pairs can satisfy (2.138). For example, constant offsets may be added to both λ and V (nonnegative only for λ) without affecting the validity of the inequality. This is expressed in the following lemma (the proof is by inspection).

Lemma 2.7.2 *Suppose that system Σ is power dissipative with power bias / storage function pair (λ, V). Then, system Σ is also power dissipative with the pair $(\lambda + L, V + M)$ for any finite nonnegative constant L and any finite constant M.*

Since the objective of the power generalization of energy dissipative systems is to facilitate the treatment of system with $\mathcal{FP}$-gain, it is imperative that power dissipative systems also exhibit $\mathcal{FP}$-gain.

Theorem 2.7.3 *Suppose that system Σ is power dissipative for gain γ with power bias / storage function pair (λ, V). Then, system Σ has $\mathcal{FP}$-gain $\leq \gamma$, and*

$$\int_0^T |z(s)|^2 \, ds \;\; \leq \;\; \gamma^2 \int_0^T |v(s)|^2 \, ds + \lambda T + V(x)$$

Proof: By definition of power dissipativeness, the pair (λ, V) must be nonnegative, finite, and satisfy the dissipation inequality (2.138). Hence, for all $v \in \mathcal{L}_2[0,T]$ and all $T \geq 0$,

$$V(x) \geq \int_0^T \left[|z(s)|^2 - \gamma^2|v(s)|^2 - \lambda\right] ds + V(x(T)).$$

But, $V(x(T)) \geq 0$, so that

$$V(x) \geq \int_0^T \left[|z(s)|^2 - \gamma^2|v(s)|^2 - \lambda\right] ds$$

for all $v \in \mathcal{L}_2[0,T]$ and all $T \geq 0$, which by Definition 2.4.2 of power gain completes the proof. ∎

It is also possible to prove that any system with $\mathcal{FP}$-gain $\leq \gamma$ is also power dissipative for gain γ. However, as this argument requires the use of a special storage function called the λ-storage, this argument is postponed until Section 2.9. However, even without this link, it is still possible to prove stability of power dissipative systems in the absence of disturbances, using techniques similar to the proof of Theorem 2.4.7.

Theorem 2.7.4 *Consider a power dissipative system Σ, with finite power bias / storage function pair (λ, V), where V is continuous. Suppose that assumptions (A5), (A14) and (A15) hold. Given any $\delta > 0$, define the compact set*

$$M_\delta = \{x \in \mathbf{R}^n : c(x) \leq \lambda + \delta\}. \tag{2.139}$$

Then, the largest zero set of the normalized storage function $\overline{V}$ is contained within the set M_0, where $M_0 = M_\delta|_{\delta=0}$ and $\overline{V}(x) = V(x) - \inf_{x\in\mathbf{R}^n}\{V(x)\}$. That is,

$$D = \left\{x \in \mathbf{R}^n : \overline{V}(x) = 0\right\} \subseteq M_0. \tag{2.140}$$

Furthermore, all trajectories of the unperturbed system tend to a compact set $M'_\delta \supseteq M_\delta$.

Proof: Given any $\delta > 0$, assumptions (A14) and (A15) implies that the set M_δ (2.139) is compact.

To prove that any trajectory must enter the set M_δ, suppose that $x \notin M_\delta$ and that there exists no $s_1 \geq 0$ such that $x(s_1) \in M_\delta$. Then, by definition of the set M_δ, clearly $c(x(s)) > \lambda + \delta$ for all $s \geq 0$. Hence, for any $T \geq 0$,

$$\int_0^T [c(x(s)) - \lambda]\, ds > \delta T. \tag{2.141}$$

However, since Σ is power dissipative with power bias / storage function pair (λ, V), V must be nonnegative, finite. Hence, the normalization $\overline{V}$ is defined. Furthermore, Lemma 2.7.2 implies that

$$\overline{V}(x) + \int_0^T \left[\gamma^2 |v(s)|^2 - c(x(s)) + \lambda\right] \, ds \;\; \geq \;\; \overline{V}(x(T)), \tag{2.142}$$

for all $v \in \mathcal{L}_2[0,T]$ and all $T \geq 0$. Choosing $v \equiv 0$ and combining (2.141) and (2.142) implies that

$$\overline{V}(x) - \delta T \;\; > \;\; \overline{V}(x(T)) \tag{2.143}$$

for any $T \geq 0$. Hence, as $\overline{V}(x)$ is finite, choosing $T \geq \frac{\overline{V}(x)}{\delta}$ implies that $\overline{V}(x) < 0$. But, the normalization $\overline{V}$ is by definition nonnegative, yielding a contradiction. That is, there exists an $s_1 \geq 0$ such that $x(s_1) \in M_\delta$.

Now suppose that $\overline{V}(x) = 0$ for some $x \notin M_\delta$. Then, applying the above argument, any $T > \frac{\overline{V}(x)}{\delta} = 0$ implies that $\overline{V}(x) < 0$, which is again a contradiction for any $\delta > 0$ arbitrarily small. Hence, $x \in \cap_{\delta > 0} M_\delta = M_0$. That is, $D \subseteq M_0$.

Finally, suppose that the trajectory starts in M_δ but exits. That is, $x \in \partial M_\delta$ and $x(s) \notin M_\delta$ for any $s > 0$ arbitrarily small. Define $B = \sup_{x \in \partial M_\delta} \left\{ \overline{V}(x) \right\}$. Since M_δ is compact and $\overline{V}$ is continuous (by assumption), clearly $B < \infty$. Hence, applying (2.141) for $T > 0$,

$$B - \delta T \;\; \geq \;\; \overline{V}(x(T)).$$

But, by the same contradiction argument, there exists an $s_1 \in (0, \frac{B}{\delta}]$ such that $x(s_1) \in M_\delta$. Hence, excursions from the set M_δ are limited in duration to $\frac{B}{\delta}$. Furthermore, these excursion are limited to a distance L from M_δ, where

$$\begin{aligned}
L \;\; &= \;\; \sup_{x \in \partial M_\delta} \sup_{t \in (0, \frac{B}{\delta}]} \inf_{y \in M_\delta} \left\{ |x(t) - y| : x(0) = x \right\} \\
&\leq \;\; \sup_{x \in \partial M_\delta} \sup_{t \in (0, \frac{B}{\delta}]} \left\{ |x(t) - x| : x(0) = x \right\} \\
&\leq \;\; \sup_{x \in \partial M_\delta} \sup_{t \in (0, \frac{B}{\delta}]} \left\{ \int_0^t |a(x(s))| \, ds : x(0) = x \right\} \\
&\leq \;\; \frac{B}{\delta} \sup_{x \in \partial M_\delta} \sup_{s \in (0, \frac{B}{\delta}]} \left\{ |a(x(s))| : x(0) = x \right\} \\
&< \;\; \infty,
\end{aligned}$$

since the trajectory $x(\cdot)$ is continuous (assumption (A5)). Hence, defining the compact set

$$M'_\delta \;\; = \;\; \left\{ x \in \mathbf{R}^n : \inf_{y \in \partial M_\delta} |x - y| \leq L \right\} \cup M_\delta, \tag{2.144}$$

any trajectory originating in M_δ is confined to M'_δ for all time. Furthermore, since we have already established that all trajectories must enter M_δ, clearly all trajectories must be eventually confined to the compact set M'_δ. ■

Remark 2.7.5 Note that Theorem 2.7.4 may be generalized to include discontinuous storage functions by considering the lower semicontinuous V_* envelope of storage function V and noting that Lemma 2.7.8 implies that V_* is also a storage function. ◀

In $\mathcal{L}_2$-gain analysis, detectability of an energy dissipative system implies that any candidate storage function for that system must have a minimum at the stable equilibrium defined to be the origin. However, for power dissipative systems, detectability implies only that the minima of storage functions be confined to a compact set, as shown in the following corollary.

Corollary 2.7.6 *Suppose that detectability assumptions (A14) and (A15) hold, and that system Σ is power dissipative. Then, any lower semicontinuous storage function V corresponding to power bias $\lambda \le \overline{\lambda}$ for system Σ has a minimum in the compact set*

$$\overline{M}_0 \quad := \quad \left\{x \in \mathbf{R}^n : c(x) \le \overline{\lambda}\right\}. \tag{2.145}$$

That is,

$$\underset{x\in\mathbf{R}^n}{\operatorname{argmin}}\{V(x)\} \subseteq \overline{M}_0. \tag{2.146}$$

Proof: Since (λ, V) is a power bias / storage function pair for Σ, V is finite and non-negative. Hence, the normalization $\overline{V}$ exists and is finite, and satisfies $\inf_{x\in\mathbf{R}^n}\left\{\overline{V}(x)\right\} = 0$. Furthermore, since V is lower semicontinuous, the set $D := \left\{x \in \mathbf{R}^n : \overline{V}(x) = 0\right\}$ is nonempty. Now, by Theorem 2.7.4 and Remark 2.7.5, $D \subseteq M_0$, where

$$M_0 \quad = \quad \{x \in \mathbf{R}^n : c(x) \le \lambda\}.$$

But, $\lambda \le \overline{\lambda}$. So, $x \in M_0$ implies that $c(x) \le \lambda \le \overline{\lambda}$. That is, $x \in \overline{M}_0$. Hence, $D \subseteq M_0 \subseteq \overline{M}_0$. ■

One of the assumptions of the Corollary 2.7.6 is lower semicontinuity of the storage function V. Since storage functions may be discontinuous, it is useful to determine under what conditions continuity (and hence lower semicontinuity) is guaranteed.

In the following proposition, the notion of local uniform reachability of the state space is shown to be a sufficient condition for the continuity of any storage function.

Proposition 2.7.7 *Suppose that assumption (A4) holds for system Σ and that (λ, V) is a finite nonnegative power bias / storage function pair for Σ. Then, the storage function $V(x)$ is continuous.*

Proof: This proof is a generalization of the proof of Proposition 1.1 of [2].

Suppose that V has a discontinuity at $x \in \mathbf{R}^n$. Applying assumption (A4) (local uniform reachability) at x', there exists a $\delta > 0$ such that (2.2) holds for all $x'' \in B(x', \delta)$. Also, since there is a discontinuity at x', there exists a sequence $\{x_n\}$ where $x_n \in B(x', \delta)$ for all n, $\lim_{n\to\infty} x_n = x'$, and $V(x_n) - V(x') \geq \varepsilon$ for some $\varepsilon > 0$. From assumption (A4), there exists an input $v_n \in \mathcal{L}_2[0, T_n]$ such that $\varphi(T_n, 0, x'; v_n) = x_n$.

Now, by definition, V satisfies the dissipation inequality, so that

$$V(x') + \int_0^{T_n} \left[\gamma^2 |v_n(s)|^2 - |z_n(s)|^2 + \lambda\right] \, ds \;\; \geq \;\; V(x_n).$$

Hence,

$$\begin{aligned} \varepsilon \leq V(x_n) - V(x') \;\; &\leq \;\; \int_0^{T_n} \left[\gamma^2 |v_n(s)|^2 + \lambda\right] \, ds \\ &= \;\; \gamma^2 \|v_n\|^2_{\mathcal{L}_2[0,T_n]} + \lambda T_n \\ &\leq \;\; \gamma^2 \alpha_1(|x_n - x'|) + \lambda \alpha_2(|x_n - x'|). \end{aligned}$$

As $n \to \infty$, since $\lambda < \infty$, clearly the right hand side of the above inequality tends to zero, which yields a contradiction since $\varepsilon > 0$.

Following a similar argument for the case where $V(x_n) - V(x') \leq -\varepsilon$ (by choosing v_n such that $\varphi(0, T_n, x_n; v_n) = x'$, $T_n < 0$), the corresponding contradiction is obtained.

Thus, V must be continuous for all $x' \in \mathbf{R}^n$. ∎

As with energy dissipativity, a perceived problem with the notion of power dissipativity is the difficulty of finding a solution of the dissipation inequality (2.138) which renders a system power dissipative. Principally, this difficulty arises from the integral form of (2.138). In the following lemma and theorem, a differential form the dissipation inequality is presented. As in energy dissipative systems theory, this differential form provides a verification result for power dissipativity. The proofs follow through in much the same way as in the energy dissipative case [24].

Lemma 2.7.8 *Suppose that (λ, V) satifies the dissipation inequality. Then, (λ, V_*) also satisfies the dissipation inequality, where V_* is the lower semicontinuous envelope of V.*

Theorem 2.7.9 *Suppose there exists a finite nonnegative viscosity solution pair (λ, V) of the partial differential inequality (PDI)*

$$\sup_{v\in\mathbf{R}^q} \left\{\nabla_x V(x)\cdot[a(x)+b(x)v]+c(x)-\gamma^2|v|^2\right\} \quad \leq \quad \lambda \tag{2.147}$$

for all $x \in \mathbf{R}^n$, where $\lambda \in \mathbf{R}^+$ and $V : \mathbf{R}^n \to \mathbf{R}^+$. Then, system Σ is power dissipative with power bias / storage function pair (λ, V_*), where V_* is the lower semicontinuous envelope of V.

Conversely, suppose that system Σ is power dissipative with power bias / storage function pair (λ, V). Then, (λ, V) is a solution pair of the PDI (2.147).

Using verification Theorem 2.7.9, it is now possible to demonstrate that a large class of nonlinear systems are power dissipative with quadratic storage functions.

Theorem 2.7.10 *Suppose that assumptions (A7), (A10), and (A12) hold. Then, there exists finite $\bar\beta > 0$ such that for any $\beta > \bar\beta$, there exists a finite $\bar\gamma_\beta$ for which system Σ is power dissipative for all gains $\gamma \geq \bar\gamma_\beta$ with power bias / storage function pair (λ_β, Q_β), where*

$$\begin{aligned} \lambda_\beta &= \beta C_2 + L_5, \\ Q_\beta &= \frac{\beta}{2}|x|^2, \end{aligned}$$

and $\bar\beta = \frac{L_5}{C_1}$, $\bar\gamma_\beta = \sqrt{\frac{L_3^2\beta^2}{4C_1(\beta-\bar\beta)}}$.

Proof: By completion of squares, the LHS of the PDI (2.147) may be written as

$$\ell_V(x) \quad := \quad \nabla_x V(x)\cdot a(x) + \frac{1}{4\gamma^2}\nabla_x V(x) b(x) b(x)' V(x)' + c(x).$$

Define $Q_\beta(x) := \frac{\beta}{2}|x|^2$, $\beta > 0$. Then,

$$\ell_{Q_\beta}(x) \quad := \quad \beta x' a(x) + \frac{\beta^2}{4\gamma^2} x b(x) b(x)' x' + c(x).$$

Applying the assumptions (A7), (A10), and (A12),

$$\begin{aligned} \ell_{Q_\beta}(x) &\leq \beta\left[-C_1|x|^2 + C_2\right] + \frac{L_3^2\beta^2}{4\gamma^2}|x|^2 + L_5(1+|x|^2) \\ &= -\left(\beta C_1 - L_5 - \frac{L_3^2\beta^2}{4\gamma^2}\right)|x|^2 + \beta C_2 + L_5 \end{aligned} \tag{2.148}$$

But,

$$\beta > \bar\beta := \frac{L_5}{C_1} \text{ and } \gamma^2 \geq \bar\gamma_\beta^2 := \frac{L_3^2\beta^2}{4C_1(\beta-\bar\beta)} \quad \Rightarrow \quad \beta C_1 - L_5 - \frac{L_3^2\beta^2}{4\gamma^2} \geq 0,$$

So, $\beta > \bar{\beta}$ and $\gamma \geq \bar{\gamma}_\beta$ implies that for all $x \in \mathbf{R}^n$,

$$\ell_{Q_\beta}(x) \leq \lambda_\beta := \beta C_2 + L_5$$

Applying Theorem 2.7.9 completes the proof. ∎

It is important to note that Theorem 2.7.10 represents a real departure from conventional energy dissipative systems theory. Quadratic storage functions for energy dissipative systems occur rarely (stable linear systems for example) whereas in the power dissipative case, it is clear that a large class of nonlinear systems have this property.

Remark 2.7.11 Referring to Theorem 2.7.10, since the minimal gain $\bar{\gamma}_\beta$ (for which system Σ is power dissipative) is parameterized by $\beta > \bar{\beta} > 0$, it is interesting to determine the choice of parameter β which minimizes $\bar{\gamma}_\beta$, as shown in Figure 2.6. Differentiating the expression for $\bar{\gamma}_\beta^2$, the minimum gain $\bar{\gamma}_\beta$ occurs when

$$\left.\frac{d\bar{\gamma}_\beta^2}{d\beta}\right|_{\beta=\beta^*} = \frac{L_3^2 c_1 \beta^*(\beta^* - 2\bar{\beta})}{4c_1^2(\beta^* - \bar{\beta})^2} = 0 \quad \Rightarrow \quad \beta^* = 2\bar{\beta}$$

Hence, the minimal upper bound on the power gain (2.22) using the growth and stability estimates of assumptions (A7), (A10), and (A12) is given by

$$\begin{aligned} \bar{\gamma}_{\beta^*} &= L_3\sqrt{\frac{\bar{\beta}}{C_1}} \\ &= \frac{L_3\sqrt{L_5}}{C_1}, \end{aligned}$$

which is precisely the power gain obtained from Theorem 2.4.8 with the maximal choice of $\delta = \frac{2C_1}{L_3}$. ◀

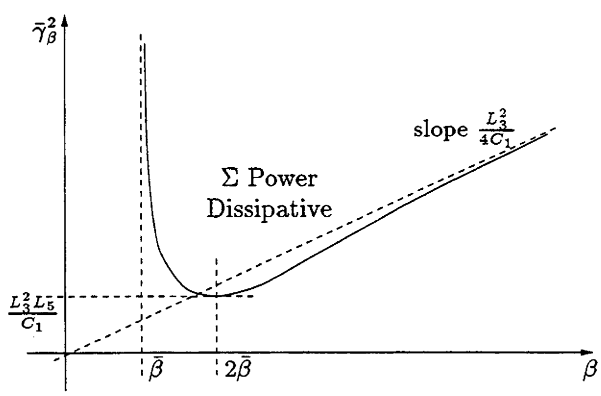


Figure 2.6: Gain $\bar{\gamma}_\beta$ versus Quadratic Coefficient β (Remark 2.7.11).

Corollary 2.7.12 *Suppose that assumptions (A7), (A10), and (A12) hold. Then, there exists a* $\bar{\gamma} \geq 0$ *such that system* Σ *has* $\mathcal{FP}$*-gain* $\leq \gamma$ *for all* $\gamma \geq \bar{\gamma}$.

Proof: Theorem 2.7.10 provides the minimal gain $\bar{\gamma} \geq 0$, and the power bias / storage function pair (λ, Q). Then apply Theorem 2.7.3 to obtain the power gain property from power dissipativeness. ∎

2.8 The Super Available Storage

With power dissipativity (Definition 2.7.1), the problem of finding a storage function V is complicated by the need to find an accompanying power bias λ. As Lemma 2.7.2 demonstrates, the power bias / storage function pair which renders a system power dissipative is not uniquely defined by the dissipation inequality (2.138).

However, in conventional dissipative systems theory we see that the available storage fulfils an important role in this respect. Not only is the available storage is the minimal storage function for energy dissipative systems, it is also defined explicitly. As such, finiteness of the available storage provides an (albeit) variational test for energy dissipativity.

So, in facing the problem of demonstrating power dissipativeness, an analogous approach requires the generalization of the notion of available storage which is meaningful for systems which exhibit internal power generation. Since the available power is a representation of the internal power generation of a system, the obvious generalization involves adding the available power λ_a to the supply rate. Since this is by no means the only generalization of available storage, the following definition is referred to as the *super available storage* (the use of the word "super" will become apparent in Section 2.10).

Definition 2.8.1 *Define the super available storage* $V_a(x)$ *as*

$$V_a(x) = \sup_{T\geq 0} \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] ds : x(0) = x \right\}. \quad (2.149)$$

Equivalently,

$$V_a(x) = \sup_{T\geq 0} \{V(x,T) - \lambda_a T\}, \quad (2.150)$$

where $V(x,T)$ *is the finite horizon value function (2.33).*

Remark 2.8.2 In view of (2.150), the super available storage may be regarded as the most stored (rather than internally generated) energy that can be retrieved from a system over any time horizon. Hence, it not suprising that the super available storage is nonnegative, as shown by choosing suboptimal $T = 0$ in (2.150). That is, $V_a(x) \geq 0$ for all $x \in \mathbf{R}^n$. ◀

As the available power / super available storage pair (λ_a, V_a) is intended as a candidate power bias / storage function pair, it is imperative that the pair satisfy the dissipation inequality (2.138). This is demonstrated in the following result, whose proof is postponed until after Theorem 2.8.11.

Theorem 2.8.3 *The available power / available storage pair (λ_a, V_a) satisfies the dissipation inequality (2.138).*

Finiteness of the available power / super available storage pair then implies power dissipativeness.

Corollary 2.8.4 *Suppose that the pair (λ_a, V_a) is finite. Then Σ is power dissipative with power bias / storage function pair (λ_a, V_a).*

Proof: Since the pair (λ_a, V_a) satisfies the dissipation inequality (2.138) by Theorem 2.8.3, finiteness (by assumption) and nonnegativity (Remark 2.8.2) of both the available power λ_a and the super available storage V_a implies by Definition 2.7.1 that the system is power dissipative with power bias / storage function pair (λ_a, V_a). ■

Clearly, the above result asserts that the available power / super available storage pair is useful in determining power dissipativity. With this in mind, we now prove some useful properties of the super available storage.

Theorem 2.8.5 *Suppose that the super available storage $V_a(x)$ (2.149) is finite. Then, the super available storage is invariant under normalization. That is,*

$$\inf_{x \in \mathbf{R}^n} \{V_a(x)\} = 0 \quad \textit{and} \quad \overline{V}_a \equiv V_a,$$

where $\overline{V}_a(x) = V_a(x) - \inf_{x \in \mathbf{R}^n}\{V_a(x)\}$ is the normalization of V_a.

Proof: By Theorem 2.8.3,

$$V_a(x) - \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] ds \;\; \geq \;\; V_a(x(T))$$

for all $v \in \mathcal{L}_2[0,T]$ and all $T \geq 0$. So, taking the inf of both sides yields that

$$V_a(x) - \sup_{T\geq 0} \sup_{v\in\mathcal{L}_2[0,T]} \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] \, ds \;\geq\; \inf_{T\geq 0} \inf_{v\in\mathcal{L}_2[0,T]} \{V_a(x(T))\}.$$

Applying the definition of V_a (2.149) (and noting that it is finite), immediately we have that

$$\begin{aligned} 0 &\geq \inf_{T\geq 0} \inf_{v\in\mathcal{L}_2[0,T]} \{V_a(x(T))\} \\ &\geq \inf_{x\in\mathbf{R}^n} \{V_a(x)\}. \end{aligned}$$

However, Remark 2.8.2 states that the opposite inequality holds. Hence,

$$0 = \inf_{x\in\mathbf{R}^n} \{V_a(x)\} \tag{2.151}$$

That is, $\overline{V}_a \equiv V_a$. ∎

In order to prove that there exists a state $\bar{x}$ for which $V_a(\bar{x}) = 0$, a lower bound for the super available storage is required.

Theorem 2.8.6 *Suppose that assumptions (A8) and (A13) hold. Then, for all $x \in \mathbf{R}^n$,*

$$V_a(x) \geq \begin{cases} R(|x|^2) & \text{if } |x|^2 \geq \rho \\ 0 & \text{if } |x|^2 < \rho \end{cases}, \tag{2.152}$$

where $R(r)$ is a strictly increasing function of $r > \rho$ given by

$$R(r) = \frac{L_6}{2C_3}(r-\rho) - \frac{L_6 M}{2C_3}\log\left(\frac{(r-\rho)+M}{M}\right), \tag{2.153}$$

$M := \frac{\lambda_a}{L_6} + \frac{C_4}{C_3} + \mu \geq 0$ and $\rho := \frac{\lambda_a}{L_6} + \mu \geq 0$.

Proof: To obtain a useful lower bound for the super available storage, choose the suboptimal disturbance $v = 0$ in the definition of V_a (2.149). That is,

$$V_a(x) \geq \sup_{T\geq 0}\left\{\int_0^T \left[c(x(s)) - \lambda_a\right] \, ds\right\}. \tag{2.154}$$

But, in the absence of disturbances,

$$\begin{aligned} x(s)'\dot{x}(s) &= x(s)'a(x(s)) \\ \Rightarrow \frac{d}{ds}\left\{|x(s)|^2\right\} &= 2x(s)'a(x(s)) \end{aligned}$$

Applying assumption (A8) and integrating,

$$\begin{aligned} \frac{d}{ds}\left\{|x(s)|^2\right\} &\geq -2C_3|x(s)|^2 - 2C_4 \\ \Rightarrow |x(s)|^2 &\geq |x|^2 e^{-2C_3 s} - \left(\frac{C_4}{C_3}\right)\left(1 - e^{-2C_3 s}\right) \end{aligned} \tag{2.155}$$

Now, combining assumption (A13) and (2.155) and integrating,

$$\begin{aligned}
\int_0^T [c(x(s)) - \lambda_a]\, ds &\geq L_6 \int_0^T |x(s)|^2\, ds - (L_6\mu + \lambda_a)T \\
&\geq L_6 \int_0^T \left[|x|^2 e^{-2C_3 s} - \left(\frac{C_4}{C_3}\right)\left(1 - e^{-2C_3 s}\right)\right] ds - (L_6\mu + \lambda_a)T \\
&= \frac{L_6}{2C_3}\left(1 - e^{-2C_3 T}\right)\left(|x|^2 + \frac{C_4}{C_3}\right) - \left[L_6\left(\frac{C_4}{C_3} + \mu\right) + \lambda_a\right] T \\
&= \frac{L_6}{2C_3}\left(1 - e^{-2C_3 T}\right)\left(|x|^2 - \rho + M\right) - L_6 M T \qquad (2.156)
\end{aligned}$$

for all $T \geq 0$, where $M := \frac{\lambda_a}{L_6} + \frac{C_4}{C_3} + \mu$ and $\rho := \frac{\lambda_a}{L_6} + \mu$. But, the RHS of (2.156) (as a function of T) has a unique stationary point when

$$e^{-2C_3 T} = \frac{M}{|x|^2 - \rho + M} \quad \text{or equivalently} \quad T = \frac{1}{2C_3}\log\left(\frac{|x|^2 - \rho + M}{M}\right),$$

which clearly corresponds to a maximum in the RHS of (2.156). In view of the objective (2.154), the maximization must be over $T \geq 0$. So, the maximizing T is then

$$T^* = \begin{cases} \frac{1}{2C_3}\log\left(\frac{|x|^2 - \rho + M}{M}\right) & \text{if } |x|^2 > \rho \\ 0 & \text{if } |x|^2 \leq \rho \end{cases}.$$

Hence, considering the nontrivial $T^* > 0$ case only, (2.156) yields that

$$\begin{aligned}
\sup_{T \geq 0}\left\{\int_0^T [c(x(s)) - \lambda_a]\, ds\right\} &\geq \frac{L_6}{2C_3}\left(|x|^2 - \rho + M - M\right) - \frac{L_6 M}{2C_3}\log\left(\frac{|x|^2 - \rho + M}{M}\right) \\
&= R(|x|^2)
\end{aligned}$$

Applying (2.154) then yields the lower bound (2.152). Monotonicity of $R(r)$ for $r > \rho$ is clear from the fact that

$$\frac{dR(r)}{dr} = \frac{L_6}{2C_3}\left(\frac{r - \rho}{r - \rho + M}\right) > 0$$

for all $r > \rho$. ∎

Remark 2.8.7 In the $\mathcal{H}_\infty$ case, where $\lambda_a = C_4 = \mu = 0$, the lower bound $R(|x|^2)$ (2.153) reduces to a simple quadratic function with minimum at the origin. ◀

Theorem 2.8.8 *Suppose assumptions (A8) and (A13) hold, and that the super available storage V_a (2.149) is finite and lower semicontinuous. Then, there exists an $\bar{x} \in \mathbf{R}^n$ such that $V_a(\bar{x}) = 0$.*

Proof: Since the super available storage V_a (2.149) is finite, Theorem 2.8.5 implies that $\inf_{x \in \mathbf{R}^n}\{V_a(x)\} = 0$. Hence, we can define a sequence x_k such that $\lim_{k \to \infty} V_a(x_k) = 0$. So, using the fact that V_a is nonnegative, given any $\varepsilon > 0$, there exists an n such that

$$k > n \quad \Rightarrow \quad 0 \le V_a(x_k) < \varepsilon.$$

Now, using assumptions (A8) and (A13), Theorem 2.8.6 states that there exists a $\rho \ge 0$ such that

$$0 < R(|x_k|^2) \le V_a(x_k) < \varepsilon$$

for all k such that $|x_k|^2 > \rho$, where R is given by (2.153). But, R is bounded on compact sets, radially unbounded, strictly increasing, and hence has a strictly increasing, radially unbounded inverse R^{-1} which is also bounded on compact sets. So,

$$k > n,\ |x_k|^2 > \rho \quad \Rightarrow \quad |x_k|^2 < R^{-1}(\varepsilon) < \infty \tag{2.157}$$

But, $\rho < R^{-1}(\varepsilon)$ by definition, so that we also get that

$$k > n,\ |x_k|^2 \le \rho \quad \Rightarrow \quad |x_k|^2 < R^{-1}(\varepsilon) < \infty \tag{2.158}$$

Hence, combining statements (2.157) and (2.158),

$$k > n \quad \Rightarrow \quad x_k \in N_\varepsilon,$$

where $N_\varepsilon := \{x \in \mathbf{R}^n : |x|^2 < R^{-1}(\varepsilon) < \infty\}$. Clearly, N_ε is closed and bounded, and hence compact. Consequently, there exists a convergent subsequence $\{x_{k_j}\}$ and an $\bar{x} \in N_\varepsilon$ such that $\lim_{j\to\infty} x_{k_j} = \bar{x}$. Furthermore, since V_a is lower semicontinuous,

$$\inf_{x\in\mathbf{R}^n} V_a(x) = \lim_{j\to\infty} V_a(x_{k_j}) = V_a(\bar{x}),$$

thereby completing the proof. ∎

Remark 2.8.9 If we define the zero set of the lower semicontinuous envelope of the finite horizon available storage V_{a*} to be $M_a = \{x \in \mathbf{R}^n : V_{a*}(x) = 0\}$, then by Theorem 2.7.4 and Theorem 2.8.8,

$$\bar{x} \in M_a \subseteq \{x \in \mathbf{R}^n : c(x) \le \lambda_a\}.$$

◀

Theorem 2.8.10 *Suppose that system Σ is power dissipative with power bias / storage function pair (λ_a, V). Then, $V(x) \ge \overline{V}(x) \ge V_a(x)$ for all $x \in \mathbf{R}^n$, where $\overline{V}(x)$ is the normalization $V(x) - \inf_{x\in\mathbf{R}^n}\{V(x)\}$.*

Proof: By definition, V must be nonnegative and must satisfy the dissipation inequality (2.138) (with corresponding power bias given by the available power λ_a).

That is,

$$\begin{aligned} V(x) &\geq \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] \, ds + V(x(T)) \\ &\geq \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] \, ds + \inf_{x\in\mathbf{R}^n} \{V(x)\} \end{aligned}$$

for all $v \in \mathcal{L}_2[0,T]$ and all $T \geq 0$. Rearranging, then taking the sup over v and T yields that

$$V(x) - \inf_{x\in\mathbf{R}^n} \{V(x)\} \;\geq\; V_a(x). \tag{2.159}$$

That is $\overline{V}(x) \geq V_a(x)$. Furthermore, since $V(x)$ is a storage function, $V(x) \geq 0$ for all $x \in \mathbf{R}^n$. Hence, $\overline{V}(x) = V(x) - \inf_{x\in\mathbf{R}^n} \{V(x)\} \leq V(x)$, completing the proof. ∎

Finally, we demonstrate that the available power / super available storage pair (λ_a, V_a) satisfies a dynamic programming equation. Using this DPE, the pair (λ_a, V_a) can be shown to solve a variational inequality [3, 37].

Theorem 2.8.11 *The available power / available storage pair (λ_a, V_a) satisfies the dynamic programming equation*

$$V_a(x) = \sup_{T\geq 0} \sup_{v\in\mathcal{L}_2[0,r\wedge T]} \left\{ \int_0^{r\wedge T} \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] \, ds + V_a(x(r))\chi_{r<T} \right\} \tag{2.160}$$

for all $r \geq 0$.

Proof: Let $r \geq 0$. From (2.150),

$$V_a(x) \geq \int_0^{r\wedge T} r(z(s), v(s)) \, ds + \int_{r\wedge T}^T r(z(s), \tilde{v}(s)) \, ds$$

where $v \in \mathcal{L}_2[0, r\wedge T]$, $\tilde{v} \in \mathcal{L}_2[r\wedge T, T]$, and $r(z,v) = |z|^2 - \gamma^2|v|^2 - \lambda$. Fix v, $r\wedge T$. Then,

$$\begin{aligned} V_a(x) &\geq \int_0^{r\wedge T} r(z(s), v(s)) \, ds + \sup_{T\geq r\wedge T, \tilde{v}\in\mathcal{L}_2[r\wedge T,T]} \left\{ \int_{r\wedge T}^T r(z(s), \tilde{v}(s)) \, ds \right\} \\ &= \int_0^{r\wedge T} r(z(s), v(s)) \, ds + \sup_{T\geq r, \tilde{v}\in\mathcal{L}_2[r,T]} \left\{ \int_r^T r(z(s), \tilde{v}(s)) \, ds \right\} \chi_{r<T} \\ &= \int_0^{r\wedge T} r(z(s), v(s)) \, ds + V(x(r))\chi_{r<T} \end{aligned} \tag{2.161}$$

for all $v \in \mathcal{L}_2[0,T]$, and all $T \geq 0$. For the opposite inequality, choose a δ-optimal stopping time $T_\delta \geq 0$ and disturbance $v_\delta \in \mathcal{L}_2[0, T_\delta]$ in (2.149). Then,

$$V_a(x) - \delta < \int_0^{T_\delta} r(z(s), v_\delta(s)) \, ds$$

$$\begin{aligned}
&= \int_0^{r\wedge T_\delta} r(z(s), v_\delta(s))\, ds + \int_{r\wedge T_\delta}^{T_\delta} r(z(s), v_\delta(s))\, ds \\
&\le \int_0^{r\wedge T_\delta} r(z(s), v_\delta(s))\, ds + \sup_{T\ge r, \tilde{v}\in\mathcal{L}_2[r,T]} \left\{ \int_r^T r(z(s), \tilde{v}(s))\, ds \right\} \chi_{r<T_\delta} \\
&= \int_0^{r\wedge T_\delta} r(z(s), v_\delta(s))\, ds + V(x(r))\chi_{r<T_\delta} \qquad (2.162)
\end{aligned}$$

Applying (2.161) for stopping time T_δ and disturbance v_δ and combining with (2.162) yields that

$$V_a(x) - \delta \;<\; \int_0^{r\wedge T_\delta} r(z(s), v_\delta(s))\, ds + V(x(r))\chi_{r<T_\delta} \le V_a(x)$$

for any $\delta > 0$. Sending $\delta \downarrow 0$ yields (2.160). ■

Note that the as yet undocumented proof of Theorem 2.8.3 forms part of the proof of the DPE (2.160). The dissipation inequality (2.138) may be recovered by choosing a suboptimal value of T in the DPE (2.160), yielding the following proof of Theorem 2.8.3.

Proof: [Theorem 2.8.3] Choose r fixed and $T > r$ in (2.160). Since T is now suboptimal,

$$V_a(x) \;\ge\; \sup_{v\in\mathcal{L}_2[0,r]} \left\{ \int_0^r \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right]\, ds + V_a(x(r)) : x(0) = x \right\},$$

which is precisely the dissipation inequality (2.138) for the pair (λ_a, V_a). ■

The following result from [37] identifies the pair (λ_a, V_a) as a solution of a variational inequality.

Theorem 2.8.12 *Suppose that the available power / super available storage pair* (λ_a, V_a) *((2.84), (2.149)) is finite. Then,* (λ_a, V_a) *is a viscosity solution pair of the variational inequality (VI)*

$$\max\left(-\lambda_a + H(x, \nabla_x V_a(x)), -V_a(x)\right) \;=\; 0 \qquad (2.163)$$

2.9 The Super λ-Storage

In energy dissipative systems theory, the available storage plays a vital role in linking $\mathcal{L}_2$-gain and energy dissipativeness. Hence, it seems reasonable that the available power / super available storage pair (λ_a, V_a) should play the same role for power dissipative systems. However, the problem which arises is that the super available storage need not be finite even with the available power finite (see for example Section 5.5). However, this can be overcome by considering a further generalization of the super available storage, called the *super λ-storage.*

Definition 2.9.1 *Define the super λ-storage $V_{a\lambda}(x)$ as*

$$V_{a\lambda}(x) = \sup_{T\geq 0} \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda\right] ds \right\}. \tag{2.164}$$

Equivalently,

$$V_{a\lambda}(x) = \sup_{T\geq 0} \{V(x,T) - \lambda T\}, \tag{2.165}$$

where $V(x,T)$ is the finite horizon value function (2.33).

Theorem 2.9.2 *Suppose that the available power λ_a (2.84) is finite. Then,*

$$\begin{aligned} \lambda > \lambda_a &\Rightarrow V_{a\lambda}(x) \leq V_a(x) \\ \lambda = \lambda_a &\Rightarrow V_{a\lambda}(x) = V_a(x) \\ \lambda < \lambda_a &\Rightarrow V_{a\lambda}(x) = \infty \end{aligned}$$

Proof: From the definition of $V_{a\lambda}$ (2.165),

$$\begin{aligned} V_{a\lambda}(x) &= \sup_{T\geq 0} \{V(x,T) - \lambda_a T + (\lambda_a - \lambda)T\} \\ &\leq \sup_{T\geq 0} \{V(x,T) - \lambda_a T\} + \sup_{T\geq 0} \{(\lambda_a - \lambda)T\} \\ &= V_a(x) + \sup_{T\geq 0} \{(\lambda_a - \lambda)T\} \\ &= V_a(x) \end{aligned}$$

if $\lambda > \lambda_a$. Similarly,

$$\begin{aligned} V_{a\lambda}(x) &\geq \sup_{T\geq 0} \{V(x,T) - \lambda_a T\} + \inf_{T\geq 0} \{(\lambda_a - \lambda)T\} \\ &= V_a(x) + \inf_{T\geq 0} \{(\lambda_a - \lambda)T\} \\ &= \infty \end{aligned}$$

if $\lambda < \lambda_a$. The $\lambda = \lambda_a$ case is trivial. ∎

Theorem 2.9.3 *System Σ has $\mathcal{FP}$-gain $\leq \gamma$ iff there exists a finite $\lambda \geq 0$ such that the super λ-storage $V_{a\lambda}(x)$ is finite for all $x \in \mathbf{R}^n$.*

Proof: Necessity: suppose that Σ has $\mathcal{FP}$-gain $\leq \gamma$ with power bias / energy bias pair (λ, β). Then, by Theorem 2.5.4,

$$V(x,T) - \lambda T \leq \beta(x)$$

for all $T \geq 0$, and all $x \in \mathbf{R}^n$. Since β is finite, taking the sup over $T \geq 0$ yields finiteness of $V_{a\lambda}$, via (2.165).

Sufficiency: taking any suboptimal $T \geq 0$ and $v \in \mathcal{L}_2[0,T]$ in (2.164) yields the inequality

$$\int_0^T |z(s)|^2\, ds \;\leq\; \gamma^2 \int_0^T |v(s)|^2\, ds + \lambda T + V_{a\lambda}(x)$$

Noting that the pair $(\lambda, V_{a\lambda})$ is finite completes the proof. ■

Lemma 2.9.4 *Given a power bias λ, the power bias / super λ-storage pair $(\lambda, V_{a\lambda})$ (2.164) satisfies the dissipation inequality (2.138). That is,*

$$V_{a\lambda}(x) + \int_0^T \left[\gamma^2 |v(s)|^2 - c(x(s)) + \lambda\right]\, ds \;\geq\; V_{a\lambda}(x(T)), \tag{2.166}$$

for all $v \in \mathcal{L}_2[0,T]$, all $T \geq 0$, and all $x \in \mathbf{R}^n$.

Proof: Let $v \in \mathcal{L}_2[0,r]$ and $\tilde{v} \in \mathcal{L}_2[r,T]$, where $r \geq 0$ is fixed and $T \geq r$. Then, by definition of the super λ-storage $V_{a\lambda}$ (2.164), the concatenation of v and $\tilde{v}$ on interval $[0,T]$ is suboptimal. That is,

$$V_{a\lambda}(x) \;\geq\; \int_0^r \left[c(x(s)) - \gamma^2 |v(s)|^2 - \lambda\right]\, ds + \int_r^T \left[c(\tilde{x}(s)) - \gamma^2 |\tilde{v}(s)|^2 - \lambda\right]\, ds,$$

where $x(0) = x$. Since this inequality holds for any choice of $\tilde{v} \in \mathcal{L}_2[r,T]$ and any $T \geq r$,

$$\begin{aligned} V_{a\lambda}(x) \;&\geq\; \int_0^r \left[c(x(s)) - \gamma^2 |v(s)|^2 - \lambda\right]\, ds + \\ &\qquad \sup_{T \geq r}\, \sup_{\tilde{v} \in \mathcal{L}_2[r,T]} \left\{ \int_r^T \left[c(\tilde{x}(s)) - \gamma^2 |\tilde{v}(s)|^2 - \lambda\right]\, ds : \tilde{x}(r) = x(r) \right\} \\ &= \int_0^r \left[c(x(s)) - \gamma^2 |v(s)|^2 - \lambda\right]\, ds + V_{a\lambda}(x(r)), \end{aligned}$$

where the definition of the super λ-storage $V_{a\lambda}(x(r))$ (2.164) has been applied. However, this inequality is just the dissipation inequality (2.138), (2.166). ■

Theorem 2.9.5 *System Σ is power dissipative iff there exists a power bias $\lambda \geq 0$ such that the power bias / super λ-storage pair $(\lambda, V_{a\lambda})$ (2.164) is finite. Furthermore, the pair $(\lambda, V_{a\lambda})$ is a power bias / storage function pair.*

Proof: Suppose that system Σ is power dissipative with power bias / storage function pair (λ, V). Then, by definition of power dissipativeness (Definition 2.7.1), (λ, V) is nonnegative, finite, and satisfies the dissipation inequality (2.138). That is,

$$\begin{aligned} V(x) &\geq \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda\right] \, ds + V(x(T)) \\ &\geq \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda\right] \, ds, \end{aligned}$$

for all $v \in \mathcal{L}_2[0,T]$, all $T \geq 0$, and all $x \in \mathbf{R}^n$. Hence, taking the supremum over $v \in \mathcal{L}_2[0,T]$ and $T \geq 0$ of the RHS and applying the definition of the super λ-storage $V_{a\lambda}$ (2.164),

$$V(x) \geq V_{a\lambda}(x)$$

for all $x \in \mathbf{R}^n$. Furthermore, $V_{a\lambda}$ is nonnegative by definition (since a suboptimal choice of the finite horizon is $T = 0$). Hence, the pair $(\lambda, V_{a\lambda})$ (2.164) is finite.

Next suppose that there exists a $\lambda \geq 0$ such that the pair $(\lambda, V_{a\lambda})$ (2.164) is finite. As mentioned, $V_{a\lambda}$ is nonegative by definition, so that the pair is also nonnegative. But, applying Lemma 2.9.4, the pair $(\lambda, V_{a\lambda})$ also satisfies the dissipation inequality. Hence, by Definition 2.7.1, system Σ is power dissipative, with power bias / storage function pair $(\lambda, V_{a\lambda})$. ∎

Finally, the equivalence of power dissipativity and power gain can be established.

Theorem 2.9.6 *System Σ is power dissipative with gain γ iff system Σ has $\mathcal{FP}$-gain $\leq \gamma$.*

Proof: Suppose that system Σ is power dissipative. Applying Theorem 2.9.5, there exists a $\lambda \geq 0$ such that the pair $(\lambda, V_{a\lambda})$ is finite. So, applying Theorem 2.9.3, system Σ has $\mathcal{FP}$-gain $\leq \gamma$. Clearly, the converse argument also holds. ∎

2.10 The Infinite Horizon Available Storage

Since the finite horizon value function $V(x,T)$ (2.33) is nondecreasing (Proposition 2.5.2), the available storage for conventional energy dissipative systems may be defined equivalently using either a supremum over $T \geq 0$, or a limit as $T \to \infty$. That is,

$$\sup_{T\geq 0} \{V(x,T)\} = \lim_{T\to\infty} \{V(x,T)\}. \tag{2.167}$$

However, for systems with nonzero available power, monotonicity of the maximum (non-generated) energy retrievable $V(x,T) - \lambda_a T$ is not assured. Hence, in the power dissipative case, (2.167) becomes

$$\sup_{T\geq 0}\{V(x,T)-\lambda_a T\} \;\geq\; \limsup_{T\to\infty}\{V(x,T)-\lambda_a T\}. \tag{2.168}$$

By Definition 2.8.1, the LHS of (2.168) is just the super available storage, which is a valid generalization of available storage for the power dissipative case. However, the RHS of (2.168) is also a possible generalization of available storage.

Definition 2.10.1 *Define the infinite horizon available storage $V_b(x)$ as*

$$V_b(x) \;=\; \limsup_{T\to\infty}\;\sup_{v\in\mathcal{L}_2[0,T]}\left\{\int_0^T \left[c(x(s))-\gamma^2|v(s)|^2-\lambda_a\right]\,ds : x(0)=x\right\}. \tag{2.169}$$

Equivalently,

$$V_b(x) \;=\; \limsup_{T\to\infty}\{V(x,T)-\lambda_a T\}, \tag{2.170}$$

where $V(x,T)$ is the finite horizon value function (2.33).

In order for the infinite horizon available storage $V_b(x)$ (2.169) to be a useful generalization in the theory of power dissipative systems, it must satisfy the dissipation inequality (2.138).

Theorem 2.10.2 *The available power / infinite horizon value function pair (λ_a, V_b) given by (2.84) and (2.169) satisifies the dissipation inequality (2.138). That is,*

$$V_b(x) \;\geq\; \sup_{v\in\mathcal{L}_2[0,r]}\left\{\int_0^r \left[c(x(s))-\gamma^2|v(s)|^2-\lambda_a\right]\,ds + V_b(x(r))\right\}, \tag{2.171}$$

for all $r\geq 0$.

Proof: Applying the definition of $V_b(x)$ (2.170) and the dynamic programming equation for $V(x,T)$ (2.40) with $0\leq r<\infty$ fixed,

$$V_b(x) \;=\; \limsup_{T\to\infty}\;\sup_{v\in\mathcal{L}_2[0,r]}\left\{\int_0^r \left[c(x(s))-\gamma^2|v(s)|^2-\lambda_a\right]\,ds \;+\; V(x(r),T-r)-\lambda_a(T-r)\right\}. \tag{2.172}$$

In order to proceed, the limsup and sup must be shown to commute. Writing $g(T,v)$ to denote the argument of the limsup sup operation in (2.172), suprema over $v\in\mathcal{L}_2[0,r]$ and $T\geq\tau$ can be shown to commute for some r, τ fixed.

$$\begin{aligned}
\sup_v\{g(T,v)\} &\geq g(T,v) \text{ for all } T,v\\
\Rightarrow \sup_T\sup_v\{g(T,v)\} &\geq \sup_T\{g(T,v)\} \text{ for all } v\\
\Rightarrow \sup_T\sup_v\{g(T,v)\} &\geq \sup_v\sup_T\{g(T,v)\}.
\end{aligned}$$

But, this argument is symmetric in T and v, so that the opposite inequality also holds. Hence, suprema over $v \in \mathcal{L}_2[0,r]$ and $T \geq \tau$ commute. However, an inf and sup commute with inequality only, since

$$\begin{aligned}
\sup_v \{g(T,v)\} &\geq g(T,v) \text{ for all } T, v \\
\Rightarrow \inf_T \sup_v \{g(T,v)\} &\geq \inf_T \{g(T,v)\} \text{ for all } v \\
\Rightarrow \inf_T \sup_v \{g(T,v)\} &\geq \sup_v \inf_T \{g(T,v)\}.
\end{aligned}$$

Hence, a limsup and sup commute with inequality only, since

$$\begin{aligned}
\limsup_{T\to\infty} \sup_{v\in\mathcal{L}_2[0,r]} \{g(T,v)\} &= \inf_{\tau\geq 0} \sup_{T\geq\tau} \sup_{v\in\mathcal{L}_2[0,r]} \{g(T,v)\} \\
&= \inf_{\tau\geq 0} \sup_{v\in\mathcal{L}_2[0,r]} \sup_{T\geq\tau} \{g(T,v)\} \\
&\geq \sup_{v\in\mathcal{L}_2[0,r]} \inf_{\tau\geq 0} \sup_{T\geq\tau} \{g(T,v)\} \\
&= \sup_{v\in\mathcal{L}_2[0,r]} \limsup_{T\to\infty} \{g(T,v)\}
\end{aligned}$$

So, applying this to (2.172),

$$\begin{aligned}
V_b(x) &\geq \sup_{v\in\mathcal{L}_2[0,r]} \limsup_{T\to\infty} \left\{ \int_0^r \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] ds + \right. \\
&\qquad\qquad \left. V(x(r), T-r) - \lambda_a(T-r) \right\} \\
&= \sup_{v\in\mathcal{L}_2[0,r]} \left\{ \int_0^r \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] ds + \right. \\
&\qquad\qquad \left. \limsup_{T\to\infty} \{V(x(r), T-r) - \lambda_a(T-r)\} \right\} \\
&= \sup_{v\in\mathcal{L}_2[0,r]} \left\{ \int_0^r \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] ds + V_b(x(r)) \right\},
\end{aligned}$$

which completes the proof. ∎

With the available power / infinite horizon available storage pair (λ_a, V_b) satisfying the dissipation inequality, (λ_a, V_b) appears a likely candidate power bias / storage function pair for testing dissipativeness. However, the next result shows that $V_b(x)$ may be negative, and so may not meet the nonnegativity requirement of Definition 2.7.1.

Corollary 2.10.3 *The infinite horizon available storage V_b (2.169) satisfies the inquality*

$$\inf_{x\in\mathbf{R}^n} \{V_b(x)\} \leq 0. \tag{2.173}$$

Furthermore, if V_b is bounded below, then $\overline{V}_b(x) \geq V_b(x)$ for all $x \in \mathbf{R}^n$, where $\overline{V}_b$ is the normalization of V_b.

Proof: By Theorem 2.10.2, V_b satisfies the dissipation inequality

$$V_b(x) - V_b(x(r)) \;\geq\; \int_0^r \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right]\, ds$$

for all $v \in \mathcal{L}_2[0,r]$ and all $r \geq 0$. Taking the sup over v and the limsup over $r \to \infty$ of both sides,

$$\begin{aligned} &V_b(x) - \liminf_{r\to\infty} \inf_{v\in\mathcal{L}_2[0,r]} \{V_b(x(r))\} \\ &\qquad \geq \; \limsup_{r\to\infty} \sup_{v\in\mathcal{L}_2[0,r]} \left\{\int_0^r \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right]\, ds\right\} = V_b(x). \end{aligned}$$

Cancelling the V_b terms,

$$\begin{aligned} 0 \;&\geq\; \liminf_{r\to\infty} \inf_{v\in\mathcal{L}_2[0,r]} \{V_b(x(r))\} \\ &\geq\; \inf_{r\geq 0}\inf_{v\in\mathcal{L}_2[0,r]} \{V_b(x(r))\} \\ &\geq\; \inf_{x\in\mathbf{R}^n} \{V_b(x)\}. \end{aligned} \tag{2.174}$$

Furthermore, with V_b bounded below, (2.174) implies that

$$\begin{aligned} \overline{V}_b(x) \;&=\; V_b(x) - \inf_{x\in\mathbf{R}^n} \{V_b(x)\} \\ &\geq\; V_b(x). \end{aligned}$$

■

The ordering $V_a(x) \geq V_b(x)$ which follows from (2.168) is formalized in the following result.

Corollary 2.10.4 *The infinite horizon available storage V_b (2.169) is bounded above by the super available storage V_a (2.149). That is, $V_b(x) \leq V_a(x)$ for all $x \in \mathbf{R}^n$.*

Proof: By Theorem 2.8.3, the pair (λ_a, V_a) satisfies the dissipation inequality (2.138). Since $V_a \geq 0$ (Remark 2.8.2),

$$\begin{aligned} V_a(x) \;&\geq\; \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right]\, ds + V_a(x(T)) \\ &\geq\; \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right]\, ds \end{aligned}$$

for all $v \in \mathcal{L}_2[0,T]$ and all $T \geq 0$. Taking the sup over v and the limsup as $T \to \infty$ completes the proof. ■

Since Theorem 2.8.10 states that the super available storage V_a is the minimal storage function for the special power bias given by the available power λ_a, Corollory 2.10.4 adds further weight to the argument that the infinite horizon available storage V_b may not be a storage function. However, the fact that $V_b(x) \leq V_a(x)$ provides another lower bound for all storage functions corresponding to the power bias λ_a.

Corollary 2.10.5 *Suppose that system Σ is power dissipative with pair (λ_a, V). Then, $V_b(x) \leq V(x)$ for all $x \in \mathbf{R}^n$.*

Proof: By Theorem 2.8.10, $V_a(x) \leq V(x)$ for all $x \in \mathbf{R}^n$. But, by Corollary 2.10.4, $V_b(x) \leq V_a(x)$ for all $x \in \mathbf{R}^n$. ∎

Corollary 2.10.6 *Suppose that the infinite horizon available storage V_b (2.169) is finite. Then, with disturbance $v_T^*(\cdot)$ optimal in the definition of the finite horizon value function $V(x,T)$ (2.33),*

$$\liminf_{T\to\infty} \{V_b(x_T^*(T))\} \leq 0, \tag{2.175}$$

where $x_T^*(\cdot)$ is the trajectory corresponding to $v_T^*(\cdot)$.

Proof: By Theorem 2.10.2 and the definition of $V(x,T)$ (2.33),

$$\begin{aligned} V_b(x) &\geq \int_0^T \left[c(x_T^*(s)) - \gamma^2|v_T^*(s)|^2 - \lambda_a\right] ds + V_b(x_T^*(T)) \\ &= \int_0^T \left[c(x_T^*(s)) - \gamma^2|v_T^*(s)|^2\right] ds - \lambda_a T + V_b(x_T^*(T)) \\ &= V(x,T) - \lambda_a T + V_b(x_T^*(T)). \end{aligned}$$

Taking the limsup as $T \to \infty$ and applying (2.170),

$$\begin{aligned} V_b(x) &\geq \limsup_{T\to\infty} \{V(x,T) - \lambda_a T\} + \liminf_{T\to\infty} \{V_b(x_T^*(T))\} \\ &= V_b(x) + \liminf_{T\to\infty} \{V_b(x_T^*(T))\}. \end{aligned}$$

Cancelling the $V_b(x)$ terms gives the required result. ∎

Provided that the infinite horizon available storage is finite and bounded below, the following result demonstrates that normalization $\overline{V}_b$ is a storage function even if the unnormalized V_b is not.

Theorem 2.10.7 *Suppose that the available power / infinite horizon available storage pair (λ_a, V_b) (given by (2.84), (2.169)) is finite and bounded below. Then, system Σ is* *power dissipative with power bias / storage function pair given by the available power and the normalized infinite horizon available storage, $(\lambda_a, \overline{V}_b)$, where*

$$\overline{V}_b(x) = V_b(x) - \inf_{x \in \mathbf{R}^n} \{V_b(x)\}. \tag{2.176}$$

Proof: By assumption, the pair (λ_a, V_b) is finite and bounded below, so that the pair $(\lambda_a, \overline{V}_b)$ is finite and nonnegative, thereby meeting the first requirement of power dissipativity. Since by Theorem 2.10.2, the pair (λ_a, V_b) satisfies the dissipation inequality (2.138), (2.171), Lemma 2.7.2 implies that the pair $(\lambda_a, \overline{V}_b)$ must also satisfy the dissipation inequality, satisfying the second requirement of power dissipativity. Hence, system Σ is power dissipative with pair $(\lambda_a, \overline{V}_b)$. ∎

This leads to the following ordering of the unnormalized and normalized definitions of available storage.

Theorem 2.10.8 *Suppose that the available power / infinite horizon available storage pair (λ_a, V_b) (given by (2.84), (2.169)) is finite and bounded below. Then, the super available storage V_a (2.149) is bounded below by the infinite horizon available storage V_b (2.169), and bounded above by the normalized infinite horizon available storage $\overline{V}_b$ (2.176). That is,*

$$V_b(x) \leq V_a(x) = \overline{V}_a(x) \leq \overline{V}_b(x) \tag{2.177}$$

for all $x \in \mathbf{R}^n$.

Proof: Applying Theorem 2.10.7, system Σ is power dissipative with pair $(\lambda_a, \overline{V}_b)$. Applying Theorem 2.8.10 provides the bound $V_a(x) \leq \overline{V}_b(x)$. Corollary 2.10.4 provides the bound $V_b(x) \leq V_a(x)$. Finally, $V_a(x) = \overline{V}_a(x)$ follows from Theorem 2.8.5. ∎

Corollary 2.10.9 *Suppose that the available power / infinite horizon available storage pair (λ_a, V_b) is finite and bounded below. Then,*

$$\inf_{x \in \mathbf{R}^n} \{\overline{V}_b(x)\} = 0$$

Furthermore, if assumptions (A8) and (A13) hold, then $\overline{V}_b$ is bounded below by the radially unbounded nonnegative function $R(|x|^2)$ (2.153). Finally, if in addition, assumption (A2) holds, and V_b is lower semicontinuous, then there exists an $\bar{x} \in \mathbf{R}^n$ such that $\overline{V}_b(\bar{x}) = 0 = V_a(\bar{x})$.

Proof: With V_b finite and bounded below, the first assertion follows directly from the definition of $\overline{V}_b$ (2.176). To prove the second assertion, apply Theorem 2.10.8. That is, $\overline{V}_b(x) \geq V_a(x)$ for all $x \in \mathbf{R}^n$. The lower bound (2.153) can then be applied. The proof of the final assertion follows similar arguments to that used in the proof of Theorem 2.8.8. Since $0 = \overline{V}_b(\bar{x}) \geq V_a(\bar{x}) \geq 0$, clearly $V_a(\bar{x}) = 0$. ∎

Remark 2.10.10 If we define the zero set of the lower semicontinuous envelope of the normalized infinite horizon available storage $\overline{V}_{b*}$ to be $M_b = \{x \in \mathbf{R}^n : \overline{V}_{b*}(x) = 0\}$, then by Theorem 2.7.4 and Corollary 2.10.9,

$$\bar{x} \in M_b \subseteq \{x \in \mathbf{R}^n : c(x) \leq \lambda_a\}.$$

◀

Since both the super available storage $V_a(x)$ (2.150) and the infinite horizon available storage $V_b(x)$ (2.170) depend explicitly on the finite horizon value function $V(x,T)$ (2.33), the finiteness of both definitions of available storage can be shown to be linked.

Theorem 2.10.11 *Suppose that assumptions (A7), (A10), and (A12) hold. Then, there exists a gain $\bar{\gamma} \geq 0$ such that for any gain $\gamma \geq \bar{\gamma}$,*

$$V_a(x) < \infty \quad \Leftrightarrow \quad V_b(x) < \infty \tag{2.178}$$

Proof: By Theorem 2.5.6, for $\gamma \geq \bar{\gamma}$, $V(x,T) \leq \lambda T + \beta |x|^2$ for all $T \geq 0$ and all $x \in \mathbf{R}^n$. But, $V(x,T) - \lambda_a T \leq (\lambda - \lambda_a) T + \beta|x|^2$, which is finite for every finite T. Hence, as $\lambda \geq \lambda_a$ (Theorem 2.6.4), $V_a(x) = \infty$ implies that the supremum in (2.150) is attained as $T \to \infty$. That is,

$$\begin{aligned} V_a(x) = \infty &= \sup_{T \geq 0} \{V(x,T) - \lambda_a T\} \\ &= \limsup_{T \to \infty} \{V(x,T) - \lambda_a T\} \\ &= V_b(x) \end{aligned}$$

Hence, $V_a(x) = \infty \Rightarrow V_b(x) = \infty$. Contrapositively, $V_b(x) < \infty \Rightarrow V_a(x) < \infty$.

For the opposite direction, Corollary 2.10.4 states that $V_b(x) \leq V_a(x)$. Hence, $V_a(x) < \infty \Rightarrow V_b(x) < \infty$. ∎

It remains to be shown that the infinite horizon available storage satisfies a dynamic programming equation and corresponding stationary PDE. However, a sketch of a possible PDE result [31] is as follows.

Conjecture 2.10.12 *The infinite horizon available storage $V_b(x)$ (2.170) is a viscosity solution of the stationary PDE*

$$H(x, \nabla_x V_b(x)) \;=\; \lambda_a \tag{2.179}$$

Proof: [Sketch] We expect that a proof of the above result will be forthcoming by recognizing that $\tilde{V}(x,T) = V(x,T) - \lambda_a T$ is a solution of the nonstationary PDE

$$\lambda_a \;=\; -\frac{\partial \tilde{V}}{\partial T}(x,T) + H(x, \nabla_x \tilde{V}(x,T)), \tag{2.180}$$

where H is the Hamiltonian (2.51). By defining $\hat{V}_T(x,t) = \tilde{V}(x,t+T)$, $\hat{V}_T$ also satisifies the PDE (2.180). But, letting $T \to \infty$, $\hat{V}_T(x,0) = \tilde{V}(x,T) \to V_b(x)$ by definition (2.170) of $V_b(x)$. Provided this convergence is uniform on compact sets, stability of viscosity solutions [17] implies that the limit $V_b(x)$ also satisfies the nonstationary PDE (2.180). However, since $\frac{\partial V_b}{\partial T}(x) = 0$, PDE (2.180) reduces to the stationary PDE (2.179). ∎

Using this result, the dynamic programming equation may be obtained by integration of the PDE (2.179). That is, for any $r \geq 0$ and any $x \in \mathbf{R}^n$,

$$V_b(x) = \sup_{v \in \mathcal{L}_2[0,r]} \left\{ \int_0^r \left[c(x(s)) - \gamma^2 |v(s)|^2 - \lambda_a \right] \, ds + V_b(x(r)) : x(0) = x \right\}. \tag{2.181}$$

Alternatively, one may assume that the dynamic programming equation (2.181) holds and prove that $V_b(x)$ is a solution of (2.179) using standard viscosity techniques [32] applied to an infinite time horizon.

Conjecture 2.10.13 *Suppose that assumptions (A5) and (A9) hold, and that $V_b(x)$ satisfies the dynamic programming equation (2.181). Then, $V_b(x)$ (2.169) is a viscosity solution of the stationary PDE (2.179),*

$$H(x, \nabla_x V_b(x)) \;=\; \lambda_a.$$

Proof: [Sketch] Begin by showing that V_b is a supersolution of the PDE. With $\phi \in C^1(\mathbf{R}^n)$, suppose that $V_b - \phi$ has a local minimum at x. Then, for sufficiently small $r > 0$,

$$V_b(x(r)) - \phi(x(r)) \;\geq\; V_b(x) - \phi(x) \tag{2.182}$$

Now, the DPE (2.181) implies that for any $v \in \mathcal{L}_2[0,r]$,

$$0 \;\geq\; \int_0^r \left[c(x(s)) - \gamma^2 |v(s)|^2 - \lambda_a \right] \, ds + V_b(x(r)) - V_b(x)$$

Applying (2.182), and dividing through by r (> 0),

$$\lambda_a \geq \frac{\phi(x(r)) - \phi(x)}{r} + \frac{1}{r}\int_0^r \left[c(x(s)) - \gamma^2|v(s)|^2\right] ds \tag{2.183}$$

Choosing $v(\cdot)$ continuous and assuming Lipschitz continuity of $a(\cdot)$ (A5) and $b(\cdot)$ (A9) to obtain continuity of $x(\cdot)$, we may send $r \downarrow 0$ and apply the Fundamental Theorem of Calculus. That is,

$$\lambda_a \geq \nabla_x\phi(x) \cdot \dot{x} + c(x) - \gamma^2|v|^2$$

which is true for all $v \in \mathbf{R}^p$. Taking the sup over v yields that

$$\lambda_a \geq H(x, \nabla_x\phi(x))$$

Hence, $V_b(x)$ is a supersolution of the PDE (2.179).

Next, show that $V_b(x)$ is a subsolution of the PDE. With $\phi \in C^1(\mathbf{R}^n)$ suppose that $V_b - \phi$ attains a local maximum at x. Then, for sufficiently small $\rho > 0$,

$$V_b(x(r)) - \phi(x(r)) \leq V_b(x) - \phi(x) \tag{2.184}$$

for all $r \in [0, \rho]$. We need to demonstrate that $H(x, \nabla_x\phi(x)) \geq \lambda_a$. So, suppose not. Then, there exists a $\theta > 0$ such that

$$H(x, \nabla_x\phi(x)) - \lambda_a \leq -\theta < 0 \tag{2.185}$$

But, the DPE (2.181) for V_b together with (2.184) implies that

$$\begin{aligned}
0 &= \sup_{v\in\mathcal{L}_2[0,\rho]} \left\{ \int_0^\rho \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] ds + V_b(x(\rho)) - V_b(x) \right\} \\
&\leq \sup_{v\in\mathcal{L}_2[0,\rho]} \left\{ \int_0^\rho \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] ds + \phi(x(\rho)) - \phi(x) \right\} \\
&= \sup_{v\in\mathcal{L}_2[0,\rho]} \left\{ \int_0^\rho \left[\nabla_x\phi(x) \cdot \dot{x} + c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] ds \right\}
\end{aligned} \tag{2.186}$$

Define

$$\Gamma(x, v) = \nabla_x\phi(x) \cdot [a(x) + b(x)v] + c(x) - \gamma^2|v|^2.$$

Boundedness of near optimal trajectories implies that for sufficiently small $\delta > 0$,

$$|x(r) - x| \leq \delta \text{ for all } r \leq \rho \text{ and all near optimal } v. \tag{2.187}$$

However, with continuity of Γ, sufficiently small δ also implies that

$$\Gamma(x(r), v(r)) - \Gamma(x, v) \leq \frac{\theta}{2}$$

Adding this to the suboptimal version of (2.185),

$$\Gamma(x(r), v(r)) \leq \frac{\theta}{2} \text{ for all } r \in [0, \rho].$$

So, integrating from 0 to ρ yields that

$$\sup_{v\in\mathcal{L}_2[0,\rho]}\left\{\int_0^\rho \Gamma(x(r),v(r))\,dr\right\} \;\le\; -\frac{\rho\theta}{2}$$

which is a contradiction of (2.186). Hence, V_b must be a subsolution, and hence a viscosity solution. ∎

Without the above conjectures, a solution $V(x)$ corresponding to power bias λ_a (2.84) of the stationary PDE (2.179) may be shown to be within a constant of the infinite horizon available storage $V_b(x)$ (2.169). Although this result does not imply that the solution $V(x)$ is the infinite horizon available storage $V_b(x)$, the explicit examples of Chapter 5 indicate, at least experimentally, that $V(x) \equiv V_b(x)$.

Before presenting the theorem, the notion of worst case disturbance is formalized.

Definition 2.10.14 *Given a C^1 solution pair (λ, V) of the PDE (2.179), the worst case disturbance is defined to be*

$$v^*(x) \;=\; \frac{1}{2\gamma^2}b(x)'\nabla_x V(x)'. \tag{2.188}$$

V is a stabilizing solution if the dynamics of corresponding to (2.1), (2.188),

$$\dot{x}^*(s) \;=\; a(x^*(s)) + \frac{1}{2\gamma^2}b(x^*(s))b(x^*(s))'\nabla_x V(x^*(s))', \tag{2.189}$$
$$x^*(0) = x,$$

are stable in the sense that there exists a compact set K such that $\lim_{s\to\infty}\{x^*(s)\} \in K$ for all $x \in \mathbf{R}^n$.

Theorem 2.10.15 *Suppose that (λ, V) is a finite nonnegative Lipschitz continuous C^1 solution pair of the stationary PDE (2.179), and that the corresponding worst case disturbance $v^*(x)$ (2.188) is stabilizing in the sense of Definition 2.10.14. Then, the power bias λ is equal to the available power (and hence minimal). Furthermore, the function V is bounded to within a constant above and below by the infinite horizon available storage. That is,*

(i) $\lambda = \lambda_a$,

(ii) $V_b(x) \le V(x) \le C + V_b(x)$ *for all* $x \in \mathbf{R}^n$, *where C is a nonnegative constant.*

Proof: Since (λ, V) satisfies the PDE (2.179) with V Lipschitz continuous, Theorem 5.2 of [16] implies that (λ, V) also satisfies the dynamic programming equation (2.181).

v^* is stabilizing in the sense that for all $x \in S$ compact, there exists an $S' \supseteq S$ and a $T' \geq 0$ such that $x^*(T) \in S'$ for all $T > T'$. Since V is finite on any compact set, there exists an $C > 0$ such that $T > T'$ and $x \in S$ implies that $V(x^*(T)) \leq C$. Hence, from (2.181), $T > T'$ and $x \in S$ implies that

$$\begin{aligned} \frac{V(x)}{T} + \lambda &= \sup_{v \in \mathcal{L}_2[0,T]} \left\{ \frac{1}{T} \int_0^T \left[c(x(s)) - \gamma^2 |v(s)|^2 \right] \, ds + \frac{V(x(T))}{T} \right\} \\ &\leq \sup_{v \in \mathcal{L}_2[0,T]} \left\{ \frac{1}{T} \int_0^T \left[c(x(s)) - \gamma^2 |v(s)|^2 \right] \, ds + \frac{C}{T} \right\}. \end{aligned}$$

Letting $T \to \infty$, the definition of available power (2.84) implies that $\lambda \leq \lambda_a$. However, by Theorems 2.7.9 and 2.7.3, system Σ has $\mathcal{FP}$-gain $\leq \gamma$ with power bias λ. with power bias / storage function pair (λ, V). Theorem 2.6.4 then implies that $\lambda_a \leq \lambda$. Hence, we have that $\lambda = \lambda_a$.

So, now Σ is power dissipative with power bias / storage function pair (λ_a, V). Corollary 2.10.5 then implies that $V_b(x) \leq V(x)$ for all $x \in \mathbf{R}^n$. Furthermore, since V is a stabilizing solution in the sense described, $T > T'$ and $x \in S$ implies that

$$\begin{aligned} V(x) &\leq \sup_{v \in \mathcal{L}_2[0,T]} \left\{ C + \int_0^T \left[c(x^*(s)) - \gamma^2 |v^*(s)| - \lambda_a \right] \, ds \right\} \\ &= C + \sup_{v \in \mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2 |v(s)| - \lambda_a \right] \, ds \right\}. \end{aligned}$$

Taking the limsup as $T \to \infty$, we get the upper bound $V(x) \leq C + V_b(x)$. ∎

The following result [16] exploits Theorem 2.10.15 to determine whether the PDE solution V is differentiable at its minimum.

Theorem 2.10.16 *Suppose that (λ, V) is a viscosity solution of the PDE (2.179), and that V has a local minimum or local maximum at $\bar{x}$. Then, if $c(\bar{x}) \neq \lambda$, V cannot be differentiable at $\bar{x}$.*

Proof: Suppose that V is differentiable at $\bar{x}$. Then, $\nabla_x V(\bar{x}) = 0$. So, substituting in the PDE (2.179), $\lambda = H(\bar{x}, 0) = c(\bar{x})$, which is a contradiction. Hence, V cannot be differentiable at $\bar{x}$. ∎

Finally, we demonstrate that if the available power / infinite horizon available storage pair (λ_a, V_b) is a solution pair of the PDE (2.179), and the corresponding worst case disturbance (2.188) drives the system to an equilibrium $\bar{x}^*$, then $V_b(\bar{x}^*) \geq 0$.

Theorem 2.10.17 *Suppose that the available power / infinite horizon available storage pair* (λ_a, V_b) *((2.84), (2.169)) is a Lipschitz continuous solution pair of the PDE (2.179). Suppose also that the corresponding worst case disturbance* v^* *(2.188) is stabilizing in the sense that the worst case dynamics* x^* *(2.189) tends to an equilibrium state* $\bar{x}^*$*. Then,* $V_b(\bar{x}^*) \geq 0$*.*

Proof: Integrating the PDE (2.179) using Theorem 5.2 of [16],

$$V_b(x) = \int_0^T \left[c(x^*(s)) - \gamma^2|v^*(s)|^2 - \lambda_a\right] ds + V_b(x^*(T)), \tag{2.190}$$

for any $T \geq 0$. Since the worst case disturbance v^* is suboptimal on the interval $[0, T]$, (2.190) implies that

$$\begin{aligned} V_b(x) &\leq \sup_{v \in \mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2|v(s)|^2 - \lambda_a\right] ds \right\} + V_b(x^*(T)) \\ &= V(x,T) - \lambda_a T + V_b(x^*(T)), \end{aligned}$$

for any $T \geq 0$. Hence, as $\lim_{T\to\infty}\{x^*(T)\} = \bar{x}^*$,

$$\begin{aligned} V_b(x) &\leq \limsup_{T\to\infty}\{V(x,T) - \lambda_a T\} + \limsup_{T\to\infty}\{V_b(x^*(T))\} \\ &= V_b(x) + V_b(\bar{x}^*), \end{aligned}$$

applying the definition of the infinite horizon available storage $V_b(x)$ (2.170). That is,

$$V_b(\bar{x}^*) \geq 0.$$

■

Remark 2.10.18 In standard $\mathcal{L}_2$-gain analysis, the equilibrium of the worst case dynamics is always located at the origin. Furthermore, this equilibrium corresponds to the minimum of the available storage. However, for the power gain case, Corollary 2.10.3 implies that

$$\inf_{x \in \mathbf{R}^n}\{V_b(x)\} \leq 0,$$

whilst Theorem 2.10.17 implies that

$$V_b(\bar{x}^*) \geq 0. \tag{2.191}$$

Hence, the correspondence of the equilibrium with the available storage minimum seen in $\mathcal{L}_2$-gain analysis is no longer guarenteed. Explicit examples of this will be presented in Chapter 5. ◀

2.11 The Infinite Horizon λ-Storage

An analogous notion to the super λ-storage (2.164) may be defined for the infinite horizon.

Definition 2.11.1 *Define the infinite horizon λ-storage $V_{b\lambda}(x)$ as*

$$V_{b\lambda}(x) = \limsup_{T\to\infty} \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2 |v(s)|^2 - \lambda \right] ds \right\}. \tag{2.192}$$

Equivalently,

$$V_{b\lambda}(x) = \limsup_{T\to\infty} \{V(x,T) - \lambda T\}, \tag{2.193}$$

where $V(x,T)$ is the finite horizon value function (2.33).

Unlike the super λ-storage, (2.193) defines a delicate balance between the asymptotic average growth $\lambda_a T$ of $V(x,T)$, and $V(x,T)$ itself. This is highlighted in the following theorem.

Theorem 2.11.2 *Suppose that the pair (λ_a, V_b) given by (2.84), (2.169) is finite and bounded below. Then, the infinite horizon λ-storage $V_{b\lambda}$ (2.192) is given by*

$$V_{b\lambda}(x) = \begin{cases} -\infty & \text{if } \lambda > \lambda_a \\ V_b(x) & \text{if } \lambda = \lambda_a \\ \infty & \text{if } \lambda < \lambda_a \end{cases}.$$

Proof: From the definition of $V_{b\lambda}$ (2.193),

$$\begin{aligned} V_{b\lambda}(x) &= \limsup_{T\to\infty} \{V(x,T) - \lambda_a T + (\lambda_a - \lambda)T\} \\ &\le \limsup_{T\to\infty} \{V(x,T) - \lambda_a T\} + \limsup_{T\to\infty} \{(\lambda_a - \lambda)T\} \\ &= V_b(x) + \limsup_{T\to\infty} \{(\lambda_a - \lambda)T\} \\ &= -\infty \end{aligned}$$

if $\lambda > \lambda_a$. Similarly,

$$\begin{aligned} V_{b\lambda}(x) &\ge \limsup_{T\to\infty} \{V(x,T) - \lambda_a T\} + \liminf_{T\to\infty} \{(\lambda_a - \lambda)T\} \\ &= V_b(x) + \liminf_{T\to\infty} \{(\lambda_a - \lambda)T\} \\ &= \infty \end{aligned}$$

if $\lambda < \lambda_a$. The $\lambda = \lambda_a$ case is trivial. ∎

2.12 Minimal Energy Supply for Systems with Fixed Initial and Final States on a Finite Time Horizon

The finite horizon value function $V(x,T)$ (2.33) is fundamental in developing the concepts of available storage for systems with power gain. Similarly, a fundamental finite horizon value function $V_r^f(\xi,x,T)$ may be defined which represents the minimal energy required to transfer the state of a system from state $\xi \in \mathbf{R}^n$ to state $x \in \mathbf{R}^n$ in time T.

Definition 2.12.1 *Define the finite horizon fixed initial state required supply $V_r^f(\xi,x,T)$ as*

$$V_r^f(\xi,x,T) \;=\; \inf_{v\in\mathcal{L}_2[-T,0]} \left\{ \int_{-T}^{0} \left[\gamma^2|v(s)|^2 - c(x(s))\right]\, ds : x(-T)=\xi, x(0)=x \right\}. \tag{2.194}$$

As with the finite horizon value function $V(x,T)$ (2.33), the power gain property imposes a growth bound on $V_r^f(\xi,x,T)$ (2.194).

Theorem 2.12.2 *Suppose that system Σ has $\mathcal{FP}$-gain $\leq \gamma$ with power bias / energy bias pair $(\lambda(x),\beta(x))$. Then,*

$$V_r^f(\xi,x,T) \;\geq\; -\lambda(\xi)T - \beta(\xi) \tag{2.195}$$

Proof: By definition of power gain (2.21),

$$\int_0^T \left[\gamma^2|v(s)|^2 - c(x(s))\right]\, ds \;\geq\; -\lambda(\xi)T - \beta(\xi) \tag{2.196}$$

for all $v \in \mathcal{L}_2[0,T]$, all $T \geq 0$, and all $\xi \in \mathbf{R}^n$. Considering in particular those disturbances which yield a final state of $x(T)=x$, the above inequality implies that

$$\inf_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[\gamma^2|v(s)|^2 - c(x(s))\right]\, ds : x(0)=\xi, x(T)=x \right\} \;\geq\; -\lambda(\xi)T - \beta(\xi).$$

But, the LHS is just $V_r^f(\xi,x,T)$ with a shift of time coordinate. ∎

The value function $V_r^f(\xi,x,T)$ (2.194) may be considered as the finite horizon precursor to a definition of required supply for systems with power gain. For this reason, it is useful both for later numerical computations and for later infinite horizon analysis to show that $V_r^f(\xi,x,T)$ satisfies a dynamic programming equation.

Theorem 2.12.3 *Given $T \geq 0$, initial state $\xi \in \mathbf{R}^n$, and final state $x \in \mathbf{R}^n$, the finite horizon fixed initial state value function $V_r^f(\xi,x,T)$ given by (2.194) satisfies the* *dynamic programming equation*

$$V_r^f(\xi,x,T) = \inf_{v\in\mathcal{L}_2[-r,0]} \left\{ \int_{-r}^{0} \left[\gamma^2|v(s)|^2 - c(x(s))\right]\, ds + V_r^f(\xi,x(-r),T-r) : x(0)=x \right\} \quad (2.197)$$

for all $r \in [0,T]$.

Proof: The first step is to rewrite the two point boundary value problem defined by (2.194) in terms of two two point boundary value problems defined on intervals $[-T,-r]$ and $[-r,0]$ respectively, the first of which has a final state corresponding to the initial state of the second. This is illustrated in Figure 2.7.

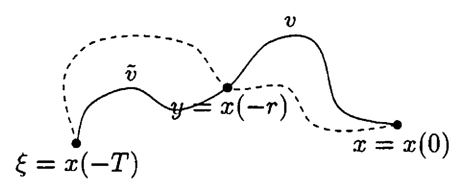


Figure 2.7: Separating the optimal trajectory by fixing an intermediary state

In view of definition (2.194) and Figure 2.7, we can rewrite (2.194) as

$$V_r^f(\xi,x,T) = \inf_{y\in\mathbf{R}^n} \inf_{v\in\mathcal{L}_2[-r,0]} \inf_{\tilde{v}\in\mathcal{L}_2[-T,-r]} \Big\{ \int_{-r}^{0} \left[\gamma^2|\tilde{v}(s)|^2 - c(\tilde{x}(s))\right]\, ds + \int_{-T}^{-r} \left[\gamma^2|v(s)|^2 - c(x(s))\right]\, ds : \tilde{x}(-T)=\xi, \tilde{x}(-r)=y=x(-r), x(0)=x \Big\}.$$

Since the two integrals are now independent (within the infimum over $y \in \mathbf{R}^n$, y is considered fixed), the first integral can be moved outside the inner most infimum over $\tilde{v} \in \mathcal{L}_2[-T,-r]$, while preserving the constraints. That is,

$$V_r^f(\xi,x,T) = \inf_{y\in\mathbf{R}^n} \inf_{v\in\mathcal{L}_2[-r,0]} \Big\{ \int_{-r}^{0} \left[\gamma^2|v(s)|^2 - c(x(s))\right]\, ds + \inf_{\tilde{v}\in\mathcal{L}_2[-T,-r]} \left\{ \int_{-T}^{-r} \left[\gamma^2|\tilde{v}(s)|^2 - c(\tilde{x}(s))\right]\, ds : \tilde{x}(-T)=\xi, \tilde{x}(-r)=y \right\} : x(-r)=y, x(0)=x \Big\}.$$

But, by definition of V_r^f (2.194), the infimum over $\tilde{v} \in \mathcal{L}_2[-T,-r]$ is just $V_r^f(\xi,y,T-r)$.

$$V_r^f(\xi,x,T) = \inf_{y\in\mathbf{R}^n} \inf_{v\in\mathcal{L}_2[-r,0]} \left\{ \int_{-r}^{0} \left[\gamma^2|v(s)|^2 - c(x(s))\right]\, ds \ + \ V_r^f(\xi,y,T-r) : \right.$$

$$x(-r) = y, x(0) = x \Big\}.$$

Finally, the infimum over $y \in \mathbf{R}^n$ relaxes the $x(-r) = y$ constraint, so that

$$V_r^f(\xi, x, T) = \inf_{v \in \mathcal{L}_2[-r,0]} \Big\{ \int_{-r}^{0} \left[\gamma^2 |v(s)|^2 - c(x(s))\right] \, ds \; + \; V_r^f(\xi, x(-r), T - r) : \\ x(0) = x \Big\},$$

which is the dynamic programming equation (2.197) for V_r^f. ∎

Finally, we propose the following theorem which associates a nonstationary PDE with the finite horizon fixed initial state required supply. Note that the proof of this theorem is straightforward in the case where $V_r^f(\xi, x, T)$ is differentiable. The viscosity proof remains outstanding however.

Theorem 2.12.4 *For a given $\xi \in \mathbf{R}^n$, suppose that the finite horizon fixed initial state required supply $V_r^f(\xi, x, T)$ (2.194) is continuous. Then, $V_r^f(\xi, x, T)$ (2.194) is a viscosity solution of the nonstationary PDE*

$$0 = -\frac{\partial V_r^f}{\partial T}(\xi, x, T) - H(x, \nabla_x V_r^f(\xi, x, T)), \tag{2.198}$$

where $H(x,p)$ is the Hamiltonian (2.51).

2.13 The Infinite Horizon Required Supply

By combining Theorems 2.6.4 and 2.12.2, it is apparent that the available power λ_a (2.84) is representative of the worst case (negative) growth rate of the finite horizon fixed initial state required supply $V_r^f(\xi, x, T)$ (2.194) with T. Hence, in defining a notion of infinite horizon required supply for systems with power gain, adding the available power to the supply rate is again an appropriate way of accounting for the internal power generation of the system.

Definition 2.13.1 *Define the infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$ as*

$$V_{br}^f(\xi, x) = \liminf_{T \to \infty} \inf_{v \in \mathcal{L}_2[-T,0]} \Big\{ \int_{-T}^{0} \left[\gamma^2 |v(s)|^2 - c(x(s)) + \lambda_a\right] \, ds : \\ x(-T) = \xi, x(0) = x \Big\}. \tag{2.199}$$

Equivalently,

$$V_{br}^f(\xi, x) = \liminf_{T\to\infty} \left\{ V_r^f(\xi, x, T) + \lambda_a T \right\}, \tag{2.200}$$

where $V_r^f(\xi, x, T)$ is the finite horizon fixed final state required supply (2.194).

In order to relate the infinite horizon available storage $V_b(x)$ (2.169) and the infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$ (2.199), it is useful to define an analogous *fixed final state* version of the infinite horizon available storage.

Definition 2.13.2 *Define the infinite horizon fixed final state available storage $V_b^f(x, \xi)$ as*

$$V_b^f(x, \xi) = \limsup_{T\to\infty} \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2 |v(s)|^2 - \lambda_a \right] ds : x(0) = x, x(T) = \xi \right\}. \tag{2.201}$$

Equivalently,

$$V_b^f(x, \xi) = \limsup_{T\to\infty} \left\{ V^f(x, \xi, T) - \lambda_a T \right\}, \tag{2.202}$$

where $V^f(x, \xi, T)$ is the finite horizon fixed final state value function (2.72).

Upon inspection of the infinite horizon fixed final state available storage $V_b^f(\xi, x)$ (2.201), intuitively we expect that the infinite horizon available storage $V_b(x)$ (2.169) can be recovered by taking the supremum of $V_b^f(x, \xi)$ over the "fixed" final state ξ. In order to prove this, the following assumption is used. The result is then stated in Lemma 2.13.3.

(A16) The limsup in (2.202) holds uniformly on compact sets with respect to the final state ξ. That is, given $\varepsilon > 0$ and $R < \infty$, there exists a τ^* such that

$$\tau > \tau^*, \ |\xi| \le R \ \Rightarrow \ 0 \le \sup_{T\ge\tau} \left\{ V^f(x, \xi, T) - \lambda_a T \right\} - V_b^f(x, \xi) < \varepsilon.$$

Lemma 2.13.3 *Suppose that assumptions (A7), (A10), (A12), and (A16) hold. Then, the infinite horizon available storage $V_b(x)$ (2.169) is the supremum over all final states $\xi \in \mathbf{R}^n$ of the infinite horizon fixed final state available storage $V_b^f(x, \xi)$ (2.201). That is,*

$$V_b(x) = \sup_{\xi\in\mathbf{R}^n} \left\{ V_b^f(x, \xi) \right\}. \tag{2.203}$$

Proof: Define an arbitrary $\rho > \bar{\rho}$, where $\bar{\rho}$ is given by (2.65), and suppose that $|x| \leq \rho$. Then, by Theorem 2.5.25,

$$V(x,T) = \sup_{|\xi| \leq R_\rho} \left\{ V^f(x,\xi,T) \right\}, \tag{2.204}$$

where R_ρ is given by (2.67). Applying assumption (A16), the limsup in (2.202) is uniform on compact sets with respect to ξ. That is, given $\varepsilon > 0$ and R_ρ (2.67), there exists a $\tau^* > 0$ such that

$$\tau > \tau^*,\ |\xi| \leq R_\rho \ \Rightarrow\ \sup_{T \geq \tau} \left\{ V^f(x,\xi,T) - \lambda_a T \right\} < V_b^f(x,\xi) + \varepsilon.$$

Hence,

$$\begin{aligned}
\tau > \tau^*,\ |x| \leq \rho \ &\Rightarrow\ \sup_{|\xi| \leq R_\rho} \sup_{T \geq \tau} \left\{ V^f(x,\xi,T) - \lambda_a T \right\} \leq \sup_{|\xi| < R_\rho} \left\{ V_b^f(x,\xi) \right\} + \varepsilon \\
&\Leftrightarrow\ \sup_{T \geq \tau} \left\{ \sup_{|\xi| \leq R_\rho} \left\{ V^f(x,\xi,T) \right\} - \lambda_a T \right\} \leq \sup_{|\xi| < R_\rho} \left\{ V_b^f(x,\xi) \right\} + \varepsilon.
\end{aligned}$$

But, applying (2.204) and noting that $\sup_{|\xi| < R_\rho} \left\{ V_b^f(x,\xi) \right\} \leq \sup_{\xi \in \mathbf{R}^n} \left\{ V_b^f(x,\xi) \right\}$,

$$\tau > \tau^*,\ |x| \leq \rho \ \Rightarrow\ \sup_{T \geq \tau} \{ V(x,T) - \lambda_a T \} \leq \sup_{\xi \in \mathbf{R}^n} \left\{ V_b^f(x,\xi) \right\} + \varepsilon.$$

That is,

$$\begin{aligned}
|x| \leq \rho \ &\Rightarrow\ \limsup_{T \to \infty} \{ V(x,T) - \lambda_a T \} \leq \sup_{\xi \in \mathbf{R}^n} \left\{ V_b^f(x,\xi) \right\} + \varepsilon \\
&\Leftrightarrow\ V_b(x) \leq \sup_{\xi \in \mathbf{R}^n} \left\{ V_b^f(x,\xi) \right\} + \varepsilon,
\end{aligned}$$

for any $\varepsilon > 0$, $\rho > \bar{\rho}$. Hence,

$$V_b(x) \leq \sup_{\xi \in \mathbf{R}^n} \left\{ V_b^f(x,\xi) \right\}. \tag{2.205}$$

But, applying Theorem 2.5.24,

$$V(x,T) = \sup_{\xi \in \mathbf{R}^n} \left\{ V^f(x,\xi,T) \right\}.$$

Subtracting $\lambda_a T$ from both sides, taking the limsup as $T \to \infty$, and applying equation (2.170) for $V_b(x)$ and equation (2.202) for $V_b^f(x,\xi)$,

$$\begin{aligned}
V_b(x) &= \limsup_{T \to \infty} \sup_{\xi \in \mathbf{R}^n} \left\{ V^f(x,\xi,T) - \lambda_a T \right\} \\
&\geq \sup_{\xi \in \mathbf{R}^n} \limsup_{T \to \infty} \left\{ V^f(x,\xi,T) - \lambda_a T \right\} \\
&= \sup_{\xi \in \mathbf{R}^n} \left\{ V_b^f(x,\xi) \right\}.
\end{aligned} \tag{2.206}$$

Combining inequalities (2.205) and (2.206) completes the proof. ∎

With this connection between the infinite horizon available storage V_b and the infinite horizon fixed final state available storage V_b^f, the aim now is to demonstrate a connection between infinite horizon fixed final state available storage V_b^f and the infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$.

Proposition 2.13.4 *The sum of the infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$ (2.199) and the infinite horizon fixed final state available storage $V_b^f(\xi, x)$ (2.201) is always zero. That is,*

$$V_{br}^f(\xi, x) + V_b^f(\xi, x) \;=\; 0 \tag{2.207}$$

for all ξ, $x \in \mathbf{R}^n$.

Proof: From the definitions of infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$ (2.199) and infinite horizon fixed final state available storage (2.201),

$$\begin{aligned}
V_{br}^f(\xi, x) &= \liminf_{T\to\infty} \inf_{v\in\mathcal{L}_2[-T,0]} \left\{ \int_0^T \left[\gamma^2|v(s-T)|^2 - c(x(s-T)) + \lambda_a\right] \, ds : \right. \\
&\qquad\qquad \left. x(-T) = \xi, x(0) = x \right\} \\
&= \liminf_{T\to\infty} \inf_{\tilde{v}\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[\gamma^2|\tilde{v}(s)|^2 - c(\tilde{x}(s)) + \lambda_a\right] \, ds : \tilde{x}(0) = \xi, \tilde{x}(T) = x \right\} \\
&= -\limsup_{T\to\infty} \sup_{\tilde{v}\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(\tilde{x}(s)) - \gamma^2|\tilde{v}(s)|^2 - \lambda_a\right] \, ds : \right. \\
&\qquad\qquad \left. \tilde{x}(0) = \xi, \tilde{x}(T) = x \right\} \\
&= -V_b^f(\xi, x).
\end{aligned}$$

■

The connection between required supply and available storage provided by Proposition 2.13.4 will be used to show that the infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$ (2.199) is a storage function. However, first we must show that $V_{br}^f(\xi, x)$ satisfies the dissipation inequality (2.138).

Theorem 2.13.5 *Suppose that the state space of Σ is completely reachable. Then, the available power / infinite horizon fixed initial state required supply pair $(\lambda_a, V_{br}^f(\xi, x))$ ((2.84), (2.199)) satisfies the dissipation inequality (2.138) with respect to x for any $\xi \in \mathbf{R}^n$ fixed.*

Proof: Applying the definition of $V_{br}^f(\xi, x)$ (2.199) and the dynamic programming equation for $V_r^f(\xi, x, T)$ (2.197) with $0 \le r < \infty$ fixed,

$$\begin{aligned}
V_{br}^f(\xi,x) &= \liminf_{T\to\infty} \inf_{v\in\mathcal{L}_2[-r,0]} \Big\{ \int_{-r}^{0} \left[\gamma^2|v(s)|^2 - c(x(s)) + \lambda_a\right] ds + \\
&\qquad V_r^f(\xi, x(-r), T-r) + \lambda_a(T-r) : x(0) = x \Big\} \\
&= \sup_{\tau\ge 0} \inf_{T\ge\tau} \inf_{v\in\mathcal{L}_2[-r,0]} \Big\{ \int_{-r}^{0} \left[\gamma^2|v(s)|^2 - c(x(s)) + \lambda_a\right] ds + \\
&\qquad V_r^f(\xi, x(-r), T-r) + \lambda_a(T-r) : x(0) = x \Big\} \\
&\le \inf_{v\in\mathcal{L}_2[-r,0]} \sup_{\tau\ge 0} \inf_{T\ge\tau} \Big\{ \int_{-r}^{0} \left[\gamma^2|v(s)|^2 - c(x(s)) + \lambda_a\right] ds + \\
&\qquad V_r^f(\xi, x(-r), T-r) + \lambda_a(T-r) : x(0) = x \Big\} \\
&= \inf_{v\in\mathcal{L}_2[-r,0]} \Big\{ \int_{-r}^{0} \left[\gamma^2|v(s)|^2 - c(x(s)) + \lambda_a\right] ds + \\
&\qquad \liminf_{T\to\infty} \Big\{ V_r^f(\xi, x(-r), T-r) + \lambda_a(T-r) \Big\} : x(0) = x \Big\} \\
&= \inf_{v\in\mathcal{L}_2[-r,0]} \Big\{ \int_{-r}^{0} \left[\gamma^2|v(s)|^2 - c(x(s)) + \lambda_a\right] ds + V_{br}^f(\xi, x(-r)) : x(0) = x \Big\},
\end{aligned}$$

where x is reachable from $x(-r)$. Subtracting $V_{br}^f(\xi, x)$ from both sides,

$$\begin{aligned}
0 \le \inf_{v\in\mathcal{L}_2[-r,0]} \Big\{ \int_{-r}^{0} \left[\gamma^2|v(s)|^2 - c(x(s)) + \lambda_a\right] ds + V_{br}^f(\xi, x(-r)) - V_{br}^f(\xi, x) : \\
x(0) = x \Big\}.
\end{aligned}$$

But, the constraint can be removed if x is chosen to be the state reached from $x(-r)$ by the application of disturbance v. That is,

$$0 \le \inf_{v\in\mathcal{L}_2[-r,0]} \Big\{ \int_{-r}^{0} \left[\gamma^2|v(s)|^2 - c(x(s)) + \lambda_a\right] ds + V_{br}^f(\xi, x(-r)) - V_{br}^f(\xi, x(0)) \Big\}.$$

So, choosing any suboptimal v yields that

$$V_{br}^f(\xi, x(-r)) + \int_{-r}^{0} \left[\gamma^2|v(s)|^2 - c(x(s)) + \lambda_a\right] ds \ \ge\ V_{br}^f(\xi, x(0)),$$

which is precisely the dissipation inequality (2.138), with power bias λ_a. ∎

Before presenting with main result of this section, the following definition is required.

Definition 2.13.6 *The state space of system Σ is defined to be reachable from state $\xi \in \mathbf{R}^n$ for all arbitrarily large times if for any $x \in \mathbf{R}^n$, there exists a $T^* < \infty$ such that for any $T \ge T^*$, there exists a $v \in \mathcal{L}_2[0, T]$ which transfers the state from state ξ to state x in time T. That is, $\varphi(T, 0, \xi; v) = x$.*

Remark 2.13.7 Although reachability in arbitrarily large times (Definition 2.13.6) may seem difficult to check, complete reachability of the state space (Definition 2.2.1) and the existence of an equilibrium for system Σ is sufficient. By labelling the equilibrium $\bar{x}$, complete reachability implies existence of a finite horizon energy disturbance v_1 which transfers the state from ξ to $\bar{x}$ in time $T_1 < \infty$. Complete reachability also implies the existence of another finite horizon energy disturbance v_2 which transfers the state from $\bar{x}$ to x in time $T_2 < \infty$. Using times T_1 and T_2, define $T^* = T_1 + T_2$ and choose any $T \geq T^*$. Applying v_1 transfers the state from ξ to $\bar{x}$. Applying no disturbance for the next $T - T^*$ time units keeps the state at $\bar{x}$. Finally, application of disturbance v_2 transfers the state to ξ, all in a total time T, which is by definition arbitrarily larger than T^*. ◀

With Remark 2.13.7 in mind, the available power / infinite horizon fixed initial state required supply pair $(\lambda_a, V_{br}^f(\xi, x))$ can be shown to be a power bias / storage function pair provided that $\xi \in \mathrm{argmin}_{x\in\mathbf{R}^n}\{V_b(x)\}$. This result may be interpreted as a generalization of Theorem 2 of [42].

Theorem 2.13.8 *Suppose that state space of Σ is reachable from state $\xi \in \mathbf{R}^n$ in all arbitrarily large times. Suppose also that the available power λ_a (2.84) is finite, and that the infinite horizon available storage V_b (2.169) is bounded below. Then, system Σ is power dissipative if there exists a constant K such that*

$$V_{br}^f(\xi, x) \;\; \geq \;\; K > -\infty \tag{2.208}$$

for all $x \in \mathbf{R}^n$. Furthermore, the pair $(\lambda_a, V_\xi(x))$ is a power bias / storage function pair, where

$$V_\xi(x) \;\; = \;\; V_b(\xi) + V_{br}^f(\xi, x). \tag{2.209}$$

Finally, suppose that $\xi \in \mathrm{argmin}_{x\in\mathbf{R}^n}\{V_b(x)\}$. Then, the pair $(\lambda_a, V_{br}^f(\xi, x))$ is also a power bias / storage function pair (with respect to x), and $V_{br}^f(\xi, x) \geq \overline{V}_b(x)$ for all $x \in \mathbf{R}^n$.

Proof: From Lemma 2.13.3, Proposition 2.13.4, and (2.208),

$$\begin{aligned} V_b(\xi) &= \sup_{x\in\mathbf{R}^n} \left\{ V_b^f(\xi, x) \right\} \\ &= -\inf_{x\in\mathbf{R}^n} \left\{ V_{br}^f(\xi, x) \right\} \end{aligned} \tag{2.210}$$

$$\leq \ -K < \infty.$$

Since it is assumed that V_b is bounded below, this upper bound implies that $V_b(\xi)$ is finite. Furthermore, reachability implies that $V_{br}^f(\xi, x)$ is also finite. But, by (2.210), $V_b(\xi) + V_{br}^f(\xi, x) \geq 0$ for all $x \in \mathbf{R}^n$. Hence, $V_\xi(x)$ (2.209) is finite and nonnegative. Furthermore, by Lemma 2.7.2 and Theorem 2.13.5, the pair $(\lambda_a, V_\xi(x))$ satisfies the dissipation inequality (2.138). Hence, system Σ is power dissipative with power bias / storage function pair $(\lambda_a, V_\xi(x))$, for any fixed $\xi \in \mathbf{R}^n$.

Applying Corollary 2.10.3 implies that with $\xi \in \operatorname{argmin}_{x\in\mathbf{R}^n}\{V_b(x)\}$, $V_b(\xi) = \inf_{x\in\mathbf{R}^n} \{V_b(x)\} \leq 0$. Hence, from (2.209),

$$\begin{aligned} V_{br}^f(\xi, x) &= V_\xi(x) - V_b(\xi) \\ &\geq V_\xi(x) \\ &\geq 0 \end{aligned}$$

That is, $V_{br}^f(\xi, x)$ is finite, nonnegative, and satisfies the dissipation inequality (2.138), and so $(\lambda_a, V_{br}^f(\xi, x))$, $\xi \in \operatorname{argmin}_{x\in\mathbf{R}^n}\{V_b(x)\}$, is a power bias / storage function pair for power dissipative system Σ.

Finally, by Theorem 2.10.2, (λ_a, V_b) satisfies the dissipation inequality (2.171). In particular,

$$\int_0^T \left[\gamma^2|v(s)|^2 - c(\xi(s)) + \lambda_a\right] \, ds \ \geq \ V_b(\xi(T)) - V_b(\xi)$$

for all $\xi \in \operatorname{argmin}_{x\in\mathbf{R}^n}\{V_b(x)\}$, all $v \in \mathcal{L}_2[0,T]$, and all $T \geq 0$. Hence,

$$\liminf_{T\to\infty} \inf_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[\gamma^2|v(s)|^2 - c(\xi(s)) + \lambda_a\right] \, ds : \xi(0) = \xi, \xi(T) = x \right\} \geq V_b(x) - V_b(\xi).$$

But, the LHS is just $V_{br}^f(\xi, x)$, while the RHS is $\overline{V}_b(x)$ (since $\xi \in \operatorname{argmin}_{x\in\mathbf{R}^n}\{V_b(x)\}$), thereby completing the proof. ∎

Note that the last assertion of Theorem 2.13.8 generalizes to include any storage function $V(x)$ corresponding to the power bias λ_a.

Theorem 2.13.9 *Suppose that system Σ is power dissipative with power bias / storage function pair $(\lambda_a, V(x))$. Suppose also that $V(x)$ attains its minimum at $\xi \in \mathbf{R}^n$. Then, $V_{br}^f(\xi, x) \geq \overline{V}(x)$ for all $x \in \mathbf{R}^n$.*

Proof: Identical to the proof of the last assertion of Theorem 2.13.8, but with V_b replaced with V. ∎

The analysis of explicit nonlinear systems with power gain (Chapter 5) reveals that typically the infinite horizon available storage $V_b(x)$ (2.169) has a unique minimum. However, in cases (as yet undiscovered) where the minimum of $V_b(x)$ in nonunique, we would like the definition of infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$ to be invariant with respect to $\xi \in \operatorname{argmin}_{x \in \mathbf{R}^n}\{V_b(x)\}$. However, this invariance with respect to initial state has yet to be proven.

Conjecture 2.13.10 *The infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$ (2.199) is invariant for all $\xi \in \operatorname{argmin}_{x \in \mathbf{R}^n}\{V_b(x)\}$. Furthermore, given $V_{br}(x) := V_{br}^f(\xi, x)$, for any $\xi \in \operatorname{argmin}_{x \in \mathbf{R}^n}\{V_b(x)\}$, the pair (λ_a, V_{br}) satisfies the PDE (2.179).*

2.14 Stability of the Worst Case Dynamics

In $\mathcal{L}_2$-gain analysis, the location of the minimum of the available storage is significant in that it corresponds to the equilibrium of the system both in absence of disturbances, and in the presence of the worst case disturbance (corresponding to the stabilizing solution of the PDE). However, Remark 2.10.18 reveals that for systems with nonzero available power, this correspondence of equilibrium with minimum may be lost. Indeed, this suspicion is confirmed for specific examples in Chapter 5.

Immediately, this draws into question the significance of both the infinite horizon available storage and the infinite horizon fixed initial state required supply with regard to the behaviour of the system in the presence of the worst case disturbance. In order that these functions provide a satisfactory generalization of the available storage / required supply for energy dissipative systems, it is very important to ascertain how the steady state worst case behaviour of the system is related to V_b and V_{br}^f. This relationship is provided by way of the following definition, assumption, and theorem.

Definition 2.14.1 *Define the W function*

$$W_\xi(x) \quad := \quad V_{br}^f(\xi, x) - V_b(x), \tag{2.211}$$

where $V_{br}^f(\xi, x)$ is the infinite horizon fixed initial state required supply (2.199), and $V_b(x)$ is the infinite horizon available storage (2.169).

(A17) The available power / infinite horizon available storage pair (λ_a, V_b) is a C^1 solution pair of the stationary PDE (2.179). Furthermore, the system (2.189) is stable in the sense of Definition 2.10.14.

Theorem 2.14.2 *Suppose that the infinite horizon available storage V_b (2.169) for system Σ is finite, bounded below, and satisfies assumption (A17). Additionally, suppose that the state space of Σ is reachable from $\xi \in \mathrm{argmin}_{x\in\mathbf{R}^n}\{V_b(x)\}$ for all arbitrarily large times (Definition 2.13.6). Then, the function $W(x)$ (2.211) is finite, bounded below, and decreases along worst case trajectories (2.189). Furthemore, the set $S_w :=$ $\mathrm{argmin}_{x\in\mathbf{R}^n}\{W_\xi(x)\}$ is invariant.*

Proof: The reachability assumption implies that $V^f_{br}(\xi, x)$ must be finite for all $x \in \mathbf{R}^n$. Since V_b is also finite, (2.211) implies that W must be finite. Furthermore, as V_b is bounded below and $\xi \in \mathrm{argmin}_{x\in\mathbf{R}^n}\{V_b(x)\}$, Theorem 2.13.8 and Corollary 2.10.3 imply that

$$V^f_{br}(\xi, x) \;\geq\; \overline{V}_b(x) \;\geq\; V_b(x)$$

for all $x \in \mathbf{R}^n$. Hence, by (2.211), $W_\xi(x) \geq 0$ for all $x \in \mathbf{R}^n$, given $\xi \in \mathrm{argmin}_{x\in\mathbf{R}^n}\{V_b(x)\}$.

Finally, we apply assumption (A17) along with Theorem 5.2 of [16]. This implies that (λ_a, V_b) is a solution pair of the dynamic programming equation (2.181),

$$\begin{aligned} &V_b(x(T)) - V_b(x) = \\ &\quad \int_0^T \left[c(x^*(s)) - \gamma^2|v^*(s)|^2 - \lambda_a\right]\, ds, \end{aligned} \tag{2.212}$$

where v^* is the worst case disturbance (2.188) (optimal in (2.179)). But, V^f_{br} satisfies the dissipation inequality (2.138) any disturbance, including v^*:

$$\begin{aligned} &V^f_{br}(\xi, x(T)) - V^f_{br}(\xi, x) \leq \\ &\quad \leq \int_0^T \left[c(x^*(s)) - \gamma^2|v^*(s)|^2 - \lambda_a\right]\, ds. \end{aligned} \tag{2.213}$$

Combining (2.212) and (2.213),

$$V^f_{br}(\xi, x(T)) - V^f_{br}(\xi, x) \;\leq\; V_b(x(T)) - V_b(x)$$

But, applying the definition (2.211), this is just

$$W_\xi(x^*(T)) \leq W_\xi(x). \tag{2.214}$$

To prove invariant of $S_w = \mathrm{argmin}_{x\in\mathbf{R}^n}\{W_\xi(x)\}$ is invariant, suppose that $x_0 \in S_w$.

Then, defining $\underline{W} = \inf_{x \in \mathbf{R}^n} \{W_\xi(x)\}$,

$$\underline{W} = W_\xi(x_0) \geq W_\xi(x(T)) \geq \underline{W}$$

Hence, $W_\xi(x(T)) = \underline{W}$ for all $T \geq 0$. That is, $x(T) \in S_w$ for all $T \geq 0$. Hence, S_w is invariant. ■

Theorem 2.14.2 reveals that for energy dissipative systems, the commonality between the location of the equilibrium and minimum of the available storage and required supply is in fact due to the commonality in the gradients (if they exist) of the available storage and the required supply. So, although for energy dissipative systems, the available storage and required supply have common gradients at the origin (which is also the minimum of both functions, and the equilibrium under the worst case disturbance), the generalization for power dissipative systems may yield V_b and V_{br}^f with common gradients away from the minimum of either function. In summary, it is the gradient of the available storage / required supply which matters, rather than the location of the minimum.

With regard to an energy flow interpretation of the W function (2.211), note that the infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$ (2.199) may be interpreted as the least energy required to transfer the state from ξ to x. Furthermore, the infinite horizon available storage may be regarded as the most energy retrievable from the system when initialized in state x. Hence, the difference $V_{br}^f(\xi, x) - V_b(x)$ is a measure of the least energy required to transfer the state from ξ to any other state in $\mathbf{R}^n$, via the state x.

Remark 2.14.3 It is important to note that the theory presented above is symmetric with respect to the required supply and available storage. That is, if we assume that V_{br}^f is an antistabilizing solution of the PDE (2.179), $W_\xi(x)$ also decreases along the worst case dynamics for the reverse time system. ◄

Chapter 3

Continuous Time Power Gain Control and Applications

3.1 Introduction

In Chapter 2, we investigated properties of nonlinear systems which exhibit power gain from disturbance to output. The systems analysed were not endowed with any internal structure apart from a simple state space description on which some assumptions regarding growth rates, etc, were made. Essentially, these systems were regarded simply as input / output maps from a disturbance to an output, as shown in Figure 3.1.

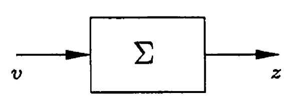


Figure 3.1: Systems Analysed in Chapter 2

An important special case of this input / output structure is a closed loop system. That is, where there is some plant G that is controlled by means of a state feedback feedback controller K, as illustrated in Figure 3.2. It is this feedback structure that will be the focus of this Chapter.

Following on naturally from Chapter 2, it will be possible to test whether a controller K endows the closed loop with the power gain property. A synthesis tool will also be developed by which a power gain controller may be derived from the plant. Furthermore, this controller K^* will be shown to be optimal in the sense that it minimizes

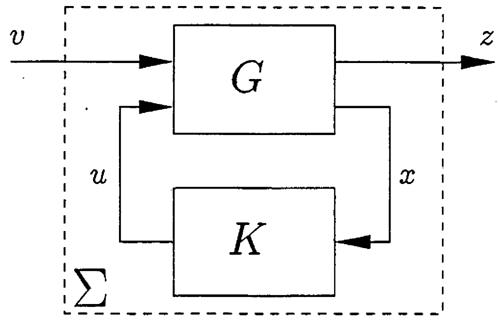


Figure 3.2: Closed Loop System $\Sigma = (G, K)$

the closed loop available power. That is,

$$\lambda_{a,K^*} = \min_{K \in \mathcal{U}_s} \{\lambda_{a,K}\}.$$

Since the available power may be regarded as the worst case power generation of a system, K^* may be interpreted as the controller which minimizes the oscillatory behaviour of the system.

To place these ideas in context, note that the standard nonlinear state feedback $\mathcal{H}_\infty$-suboptimal synthesis techniques (for example, [38]) result in a static state feedback controller K which endows the closed loop system with a prescribed maximum energy ($\mathcal{L}_2$-) gain from disturbance v to output z. As such, the energy gain property is in fact a design constraint. However, under a detectability assumption, this constraint implies directly asymptotic stability of the closed loop system Σ. In this chapter, the energy gain constraint is weakened to a power gain constraint. This relaxation in the design means that asymptotically stable behaviour of the closed loop is not the only admissible behaviour. Consequently, it becomes possible for the closed loop system to exhibit oscillatory or limit cycle behaviour. This fact forms the main motivation for this chapter.

Since the available power is a useful measure of the nonzero steady state behaviour of a system, it is also possible to naturally define an optimal power gain control problem. As already mentioned, the solution of this problem yields a prescription for not only synthesizing a controller, but synthesizing possibly the most useful controller: the one which minimizes the closed loop available power.

Finally, we present an application for power gain control. In particular, we consider the control of linear systems with actuator nonlinearities.

3.2 Class of Systems

Throughout this chapter, we consider the plant G and the controller K to be nonlinear systems of the form

$$\Sigma \; : \; \begin{cases} \dot{x} &= a(x) + b_1(x)u + b_2(x)v, \\ z &= h(x) + d(x)u, \end{cases} \tag{3.1}$$

$$K \; : \; u = u(x), \tag{3.2}$$

where $x(t) \in \mathbf{R}^n$ is the state of the system at time t, $u(t) \in \mathbf{R}^m$ is the actuating input, $v(t) \in \mathbf{R}^p$ is the disturbance, and $z(t) \in \mathbf{R}^q$ is the output.

3.3 State Feedback Power Gain Control

Definition 3.3.1 *A static controller K (3.2) is a power gain controller with gain $\leq \gamma$ if the closed loop system $\Sigma \equiv (G, K)$ has $\mathcal{FP}$-gain $\leq \gamma$ (Definition 2.4.2). That is, there exists finite nonnegative pair (λ_K, β_K) (dependent on the controller K) such that*

$$\int_0^T |z(s)|^2 \, ds \;\leq\; \gamma^2 \int_0^T |v(s)|^2 \, ds + \lambda_K T + \beta_K(x) \tag{3.3}$$

for all $v \in \mathcal{L}_2[0,T]$, all $T \geq 0$, and all $x \in \mathbf{R}^n$, where x is the initial state, v is the disturbance, and z is the output of system Σ given by (3.1), (3.2).

Theorem 3.3.2 *Suppose there exists a power gain controller K (3.2) with gain $\leq \gamma$ for open loop plant G (3.1). Then, there exists a finite nonnegative solution pair (λ_K, V_K) of the PDI*

$$H_1(x, \nabla_x V_K(x), u_K(x)) \;\leq\; \lambda_K, \tag{3.4}$$

where $u_K(x)$ is the control policy for controller K (3.2), and H_1 is the Hamiltonian

$$H_1(x,p,u) \;=\; \sup_{v \in \mathbf{R}^p} \{h(x,p,u,v)\}, \tag{3.5}$$

$$h(x,p,u,v) \;=\; p \cdot [a(x) + b_1(x)u + b_2(x)v] + |c(x) + d(x)u|^2 - \gamma^2 |v|^2 . \tag{3.6}$$

Conversely, suppose for a given controller K, there exists a finite nonnegative viscosity solution pair (λ_K, V_K) of the PDI (3.4) for gain γ. Then, the controller K is a power gain controller with gain $\leq \gamma$. Furthermore, if V_K is C^1, then the worst case disturbance for the closed loop system $\Sigma \equiv (G, K)$ with power bias / storage function pair (λ_K, V_K)

is given by

$$v_K^*(x) \;=\; \operatorname*{argmax}_{v \in \mathbf{R}^p} \{h(x, \nabla_x V_K(x), u_K(x), v)\}. \tag{3.7}$$

Proof: Suppose that controller K (3.2) is a power gain controller with $\gamma \leq \gamma$ for plant G (3.1). Then, by Definition 3.3.1, the closed loop system $\Sigma \equiv (G, K)$ has $\mathcal{FP}$-gain $\leq \gamma$. That is, there exists a finite nonegative power bias / energy bias pair (λ_K, β_K) such that the power gain inequality (3.3), (2.21) holds for gain γ. Applying Theorem 2.9.6, this implies that system $\Sigma \equiv (G, K)$ is power dissipative for gain γ, and by Theorem 2.9.5, $(\lambda_K, V_{a\lambda,K})$ is an admissible power bias / storage function pair, where $V_{a\lambda,K}$ is the super λ-storage for the closed loop system (G, K). Finally, applying Theorem 2.7.9, $(\lambda_K, V_{a\lambda,K})$ is a viscosity solution pair of the PDI (3.4).

Next, suppose that for a given controller K, there exists a finite nonegative solution pair (λ_K, V_K) of the PDI (3.4). Then, by Theorem 2.7.9, the closed loop system $\Sigma \equiv (G, K)$ must be power dissipative for gain γ, with power bias / storage function pair (λ_K, V_{K*}), where V_{K*} is the lower semicontinuous envelope of V_K. Applying Theorem 2.7.3, the closed loop system $\Sigma \equiv (G, K)$ has $\mathcal{FP}$-gain $\leq \gamma$ with power bias / energy bias pair (λ_K, V_{K*}). Hence, from Definition 3.3.1, K is a power gain controller with gain $\leq \gamma$.

Finally, suppose that V_K is C^1. Then, completion of squares in the PDI (3.4) yields (3.7). ■

Theorem 3.3.3 *Suppose there exists a power gain controller K (3.2) with gain $\leq \gamma$ for the open loop plant G (3.1). Then, there exists a finite nonegative solution pair (λ_K, V_K) of the PDI*

$$H(x, \nabla_x V_K(x)) \;\leq\; \lambda_K, \tag{3.8}$$

where

$$H(x, p) \;=\; \inf_{u \in \mathbf{R}^m} \{H_1(x, p, u)\}, \tag{3.9}$$

and H_1 is given by (3.5).

Proof: By Theorem 3.3.2, there exists a finite nonegative solution pair (λ_K, V_K) of the PDI (3.4). That is,

$$H_1(x, \nabla_x V_K(x), u_K(x)) \;\leq\; \lambda_K,$$

where $u_K(x)$ is the control policy for controller K. But,

$$\begin{aligned}\inf_{u\in\mathbf{R}^m}\{H_1(x,\nabla_x V_K(x),u)\} &\leq H_1(x,\nabla_x V_K(x),u_K(x))\\ &\leq \lambda_K.\end{aligned}$$

So, (λ_K, V_K) is also a finite nonnegative solution pair of the PDI (3.8). ∎

Definition 3.3.4 *A static state feedback controller K is defined to be admissible if the control policy $u(x)$ for K is a finite mapping from $\mathbf{R}^n$ to $\mathbf{R}^m$. The set of such controllers is defined to be $\mathcal{U}_s$.*

Definition 3.3.5 *A power gain controller K^* with gain $\leq \gamma$ is defined to be optimal if the available power of the closed loop system (G, K^*) defined by (2.84) is minimal with respect to the set of admissible controllers $\mathcal{U}_s$. That is,*

$$\lambda_{a,K^*} = \min_{K\in\mathcal{U}_s}\{\lambda_{a,K}\} =: \lambda_a^*, \tag{3.10}$$

where $\lambda_{a,K}$ is the available power for the closed loop system (G, K).

The following theorem is an application of Theorem 8.2, Corollary 8.3, and equation (8.17) of [16].

Theorem 3.3.6 *Suppose there exists a Lipschitz continuous C^1 nonnegative solution pair (λ, V) for gain γ of the PDE*

$$H(x,\nabla_x V(x)) = \lambda, \tag{3.11}$$

where H is given by (3.9). Then, the controller

$$K^* \;:\; u^*(x) = \operatorname*{argmin}_{u\in\mathbf{R}^m}\{H_1(x,\nabla_x V(x),u)\} \tag{3.12}$$

is the optimal power gain controller with gain $\leq \gamma$, in that (3.10) holds. That is, the available power of the closed loop system (G, K^*) is minimal with respect to the set of admissible controllers $\mathcal{U}_s$.

3.4 Linear Systems with Actuator Nonlinearities

3.4.1 Introduction

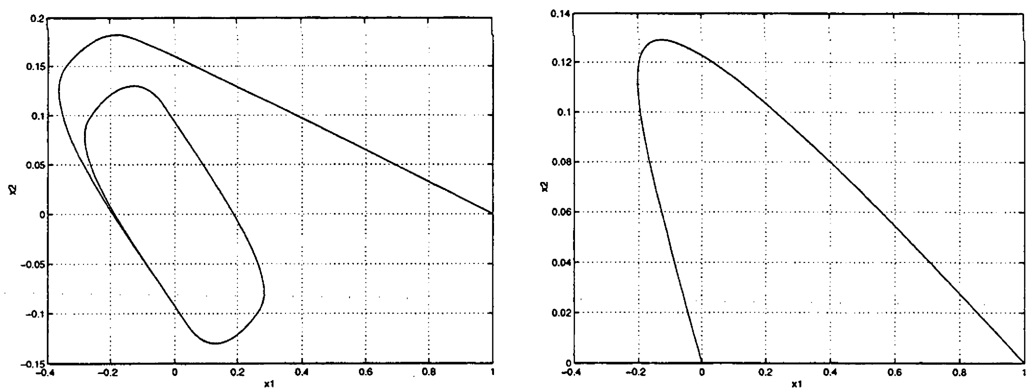
Linear systems with actuator nonlinearities can often exhibit oscillatory behaviour in closed loop. For example, consider an unstable linear plant with a deadzone actuator nonlinearity,

$$G \;:\; \begin{cases} \dot{x} \;=\; \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \alpha(u) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v, \\ z \;=\; \begin{bmatrix} x_1 \\ \alpha(u) \end{bmatrix}, \end{cases} \tag{3.13}$$

where $\alpha(\cdot)$ is a deadzone centered at 0 with deadband $[-1,1]$. By neglecting the nonlinearity and applying standard linear $\mathcal{H}_\infty$ techniques for gain $\gamma = 4$, a controller for the system is given by

$$K_L \;:\; u(x) = \begin{bmatrix} -8.14 & -11.93 \end{bmatrix} x.$$

Figure 3.3(a) illustrates that the closed loop (G, K_L) exhibits limit cycle behaviour in the absence of disturbances. So, application of controller K_L results in nonzero available power for the closed loop (G, K_L). In fact, we will show that controller K_L is a power gain controller with gain ≤ 4 for the plant G.



(a) (G, K_L) Oscillatory (b) (G, K_N) Asymptotically Stable

Figure 3.3: An Unstable Linear System with Deadzone Nonlinearity

An alternative to K_L is the nonlinear controller

$$K_N \;:\; u(x) = \alpha^{-1}\left(\begin{bmatrix} -8.14 & -11.93 \end{bmatrix} x\right).$$

Figure 3.3(b) illustrates that the closed loop (G, K_N) exhibits no limit cycle behaviour. In fact, the closed loop (G, K_N) is asymptotically stable. Hence, application of controller K_N results in zero available power for the closed loop (G, K_N). Consequently, K_N minimizes the available power for the closed loop (G, K) for any controller K, since the available power for (G, K) is bounded below by zero. That is, K_N must be the optimal power gain controller K^* (3.12) with gain ≤ 4 for the plant G. In fact, the optimality of K^N will be seen to follow directly from Theorem 3.3.6. This example demonstrates that a useful way of characterizing the oscillatory behaviour of a system is to consider the available power. By minimizing the available power of the system, we seek to minimize the nonzero steady state behaviour.

In summary, the remainder of the Chapter will be devoted to the application of results of Section 3.3 for linear plants with actuator nonlinearities. Specifically, the class of systems considered is a special case of (3.1), (3.2), with

$$G : \begin{cases} \dot{x} = Ax + B_1\tilde{u} + B_2 v, \\ z = \begin{bmatrix} Cx \\ \mu(u) \end{bmatrix}, \end{cases} \tag{3.14}$$

$$\alpha : \tilde{u} = \alpha(u), \tag{3.15}$$

$$K : u = u(x). \tag{3.16}$$

The structure of such systems is illustrated in Figure 3.4. In considering the problem

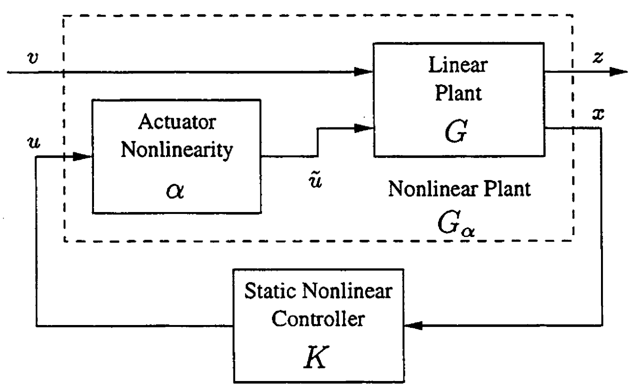


Figure 3.4: Controlling a Linear System with Actuator Nonlinearity

of controlling linear systems with actuator nonlinearities, we begin by discussing the applicability of a linear $\mathcal{H}_\infty$ control design. In particular, we find that a linear controller can often lead to an oscillatory closed loop. In contrast, the nonlinear optimal control technique described by Theorem 3.3.6 results in a nonlinear controller which seeks to invert the nonlinearity, as in [40].

3.4.2 Power Gain Control using a Linear Controller

When designing a controller for a linear plant with an actuator nonlinearity, one approach is to recast the nonlinearity as a plant uncertainty, denoted by Δ. Since the remainder of the plant is then linear, standard linear $\mathcal{H}_\infty$ control techniques can be applied, yielding a linear controller K_L. This configuration is illustrated in Figure 3.5. Note that G_L and G (Figures 3.5 and 3.4 respectively) will be not indentical.

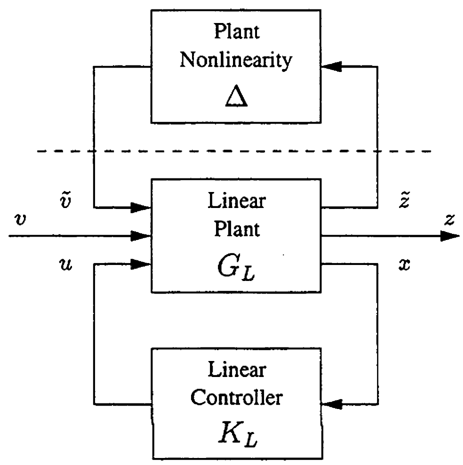


Figure 3.5: Actuator Nonlinearity recast as Plant Uncertainty

With the plant uncertainty modelled by the disturbance $\tilde{v} \in \mathbf{R}^m$, the state equation for G_α (3.14), (3.15) can be rewritten as

$$\begin{aligned} \dot{x} &= Ax + B_1\alpha(u) + B_2 v \\ &= Ax + B_1 Qu + B_1(\alpha(u) - Qu) + B_2 v \\ &= Ax + B_1 Qu + B_1\tilde{v} + B_2 v \\ &= Ax + B_1 Qu + \begin{bmatrix} B_2 & B_1 \end{bmatrix} \begin{bmatrix} v \\ \tilde{v} \end{bmatrix}. \end{aligned}$$

where Q is an invertible $m \times m$ matrix, and $\tilde{v}$ is a feedback disturbance $\Delta(\cdot)$ involving the nonlinearity $\alpha(\cdot)$. The resulting recast system can now be written fully as

$$G_L \; : \; \left\{ \begin{array}{rcl} \dot{x} &=& Ax + B_1 Qu + \begin{bmatrix} B_2 & B_1 \end{bmatrix} \begin{bmatrix} v \\ \tilde{v} \end{bmatrix}, \\ \begin{bmatrix} z \\ \tilde{z} \end{bmatrix} &=& \begin{bmatrix} Cx \\ u \end{bmatrix}, \end{array} \right. \tag{3.17}$$

$$K_L \;:\; u = Lx, \tag{3.18}$$

$$\Delta \;:\; \tilde{v} = \alpha(\tilde{z}) - Q\tilde{z}, \tag{3.19}$$

as per Figure 3.5.

Since G_L is linear, the controller K_L may be designed independently of the actuator nonlinearity Δ, using standard linear $\mathcal{H}_\infty$ techniques. That is, if there exists a positive definite solution P of the algebraic Riccati equation

$$A'P + PA + P\left(\frac{1}{\gamma_1^2}\left(B_2B_2' + B_1B_1'\right) - B_1QQ'B_1'\right)P + C'C \;=\; 0, \tag{3.20}$$

then the linear controller given by (3.18), $L = -Q'B_1'P$, yields a closed loop (G_L, K_L) with $\mathcal{L}_2$-gain $\leq \gamma_1$.

Although controller K_L endows the closed loop (G_L, K_L) with energy gain $\leq \gamma_1$, this does not necessarily imply that the controller K_L is a power gain controller for the modified system (Δ, G_L, K_L). For this, we require a simple generalization of the Small Gain Theorem [21].

Theorem 3.4.1 (Small Gain) *Suppose that system Σ_1 has $\mathcal{FP}$-gain $\leq \gamma_1$ from disturbance $\begin{bmatrix} v \\ \tilde{v} \end{bmatrix}$ to output $\begin{bmatrix} z \\ \tilde{z} \end{bmatrix}$ with power bias / energy bias pair (λ_1, β_1). Suppose also that system Σ_2 has $\mathcal{FP}$-gain $\leq \gamma_2$ from disturbance $\tilde{z}$ to output $\tilde{v}$ with power bias / energy bias pair (λ_2, β_2). Then, the feedback interconnection shown in Figure 3.6 has $\mathcal{FP}$-gain $\leq \gamma_1$ from disturbance v to output z with power bias / energy bias pair $(\lambda_1 + \gamma_1^2\lambda_2, \beta_1 + \gamma_1^2\beta_2)$ if the small gain condition*

$$\gamma_1\gamma_2 \;\leq\; 1 \tag{3.21}$$

holds.

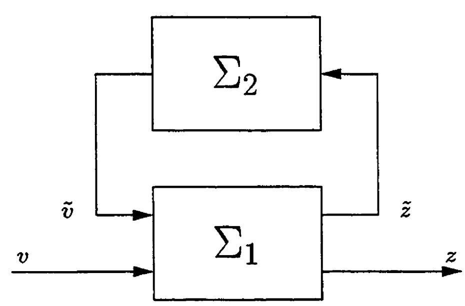


Figure 3.6: Feedback Interconnection of Two Systems

Proof: Σ_2 has $\mathcal{FP}$-gain $\leq \gamma_2$ with power bias / energy bias pair (λ_2, β_2). Hence, by Definition 2.4.2,

$$\gamma_1^2 \int_0^T |\tilde{v}(s)|^2 \, ds \;\; \leq \;\; \gamma_1^2\gamma_2^2 \int_0^T |\tilde{z}(s)|^2 \, ds + \gamma_1^2 \lambda_2 T + \gamma_1^2 \beta_2(x). \tag{3.22}$$

But, Σ_1 has $\mathcal{FP}$-gain $\leq \gamma_1$ with power bias / energy bias pair (λ_1, β_1). So,

$$\int_0^T \left[|z(s)|^2 + |\tilde{z}(s)|^2\right] \, ds \;\; \leq \;\; \gamma_1^2 \int_0^T \left[|v(s)|^2 + |\tilde{v}(s)|^2\right] \, ds + \lambda_1 T + \beta_1(x). \tag{3.23}$$

Combining (3.22) and (3.23),

$$\begin{aligned}\int_0^T \left[|z(s)|^2 + \left(1 - \gamma_1^2\gamma_2^2\right) |\tilde{z}(s)|^2\right] \, ds \;\; \leq \;\; & \gamma_1^2 \int_0^T |v(s)|^2 \, ds + \left(\lambda_1 + \gamma_1^2 \lambda_2\right) T + \\ & \left(\beta_1(x) + \gamma_1^2 \beta_2(x)\right). \end{aligned} \tag{3.24}$$

But, if the small gain condition (3.21) holds, then

$$\int_0^T |z(s)|^2 \, ds \;\; \leq \;\; \int_0^T \left[|z(s)|^2 + \left(1 - \gamma_1^2\gamma_2^2\right) |\tilde{z}(s)|^2\right] \, ds. \tag{3.25}$$

Combining (3.24) and (3.25) then yields the required power gain inequality from disturbance v to output z for the feedback interconnection of Figure 3.6. ■

This version of the Small Gain Theorem may now be applied to systems with structure of the form shown in Figure 3.5. Comparing with the feedback interconnection of Figure 3.6, plant and controller (G_L, K_L) may be lumped together to form the subsystem Σ_1, while the plant uncertainty Δ can form the subsystem Σ_2. Furthermore, since K_L was designed using $\mathcal{H}_\infty$ techniques, subsystem Σ_1 must have $\mathcal{L}_2$-gain $\leq \gamma_1$, where $\gamma_1 \geq \|(G_L, K_L)\|_{\mathcal{H}_\infty}$. Hence, $\lambda_1 = 0$. The proof of the following result (the standard Small Gain Theorem) is immediate from Theorem 3.4.1.

Corollary 3.4.2 *Suppose that the system (G_L, K_L) is detectable. Then, the system (Δ, G_L, K_L) is asymptotically stable in the absence of disturbances if there exists gains γ_1 and γ_2 such that*

$$\gamma_1 \;\; > \;\; \|(G_L, K_L)\|_{\mathcal{H}_\infty}, \tag{3.26}$$

$$\gamma_2 \;\; > \;\; \inf\left\{\gamma \geq 0 : \lambda_2^\gamma = 0\right\}, \tag{3.27}$$

and (3.21) hold simultaneously.

Theorem 3.4.3 *Suppose that the matrix C is invertible and the actuator nonlinearity $\alpha(\cdot)$ satisfies the growth condition*

$$|\alpha(u)| \;\; \leq \;\; L(1 + |u|). \tag{3.28}$$

Then, the system (Δ, G_L, K_L) is stable, in that unperturbed trajectories tend to a compact set, if there exists gains γ_1 and γ_2 such that

$$\gamma_2 \quad > \quad \inf\{\gamma \geq 0 : \lambda_2^\gamma < \infty\}, \tag{3.29}$$

and (3.26), (3.21) hold simultaneously.

Proof: Since inequalities (3.21), (3.26), and (3.29) hold simultaneously, Theorem 3.4.1 implies that the closed loop system (Δ, G_L, K_L) has $\mathcal{FP}$-gain $\leq \gamma_1$. Furthermore, the closed loop drift for system (Δ, G_L, K_L) is given by

$$a_{cl}(x) \quad = \quad Ax + B_1\alpha(Lx),$$

where $L = -Q'B_1'$. Hence, applying (3.28),

$$\begin{aligned} |a_{cl}(x)| \quad &\leq \quad |Ax| + |B_1||\alpha(Lx)| \\ &\leq \quad L_1(1 + |x|), \end{aligned}$$

for some $L_1 > 0$, which is precisely assumption (A6). Finally, since C is invertible, assumptions (A14) and (A15) hold. So, since (Δ, G_L, K_L) has $\mathcal{FP}$-gain, and assumptions (A6), (A14), and (A15) hold, Theorem 2.4.7 implies that the system (Δ, G_L, K_L) is stable in that the unperturbed trajectories (ie $v = 0$) tend to a compact set. ∎

To proceed further requires knowledge of the specific actuator nonlinearity involved, so that the gain γ_2 and the power bias λ_2 may be calculated.

Example 3.4.4 Suppose that the actuator nonlinearity is a deadzone of the form

$$\alpha(u) \quad = \quad [\alpha_1(u_1)\ \alpha_2(u_2)\ \cdots\ \alpha_m(u_m)]',$$

where

$$\alpha_i(u_i) = \begin{cases} d_{1i}u_i + d_{2i} & \text{if} \quad u_i < -\frac{d_{2i}}{d_{1i}} \\ 0 & \text{if} \quad u_i \in [-\frac{d_{2i}}{d_{1i}}, \frac{d_{2i}}{d_{1i}}] \\ d_{1i}u_i - d_{2i} & \text{if} \quad u_i > \frac{d_{2i}}{d_{1i}} \end{cases} \tag{3.30}$$

and d_{1i}, $d_{2i} > 0$. If we define $D_1 \in \mathbf{R}^{m \times m}$ as diagonal with i^{th} entry d_{1i}, and choose $Q = D_1$ in (3.19), then Δ is a saturation given by $\Delta(z) = [\Delta_1(z_1)\ \Delta_2(z_2)\ \cdots\ \Delta_m(z_m)]'$, where

$$\Delta_i(z_i) = \begin{cases} d_{2i} & \text{if} \quad z_i < -\frac{d_{2i}}{d_{1i}} \\ -d_{1i}z_i & \text{if} \quad z_i \in [-\frac{d_{2i}}{d_{1i}}, \frac{d_{2i}}{d_{1i}}] \\ -d_{2i} & \text{if} \quad z_i > \frac{d_{2i}}{d_{1i}} \end{cases} \tag{3.31}$$

Since Δ is a static nonlinearity, we follow the same steps as in Section 5.1 to calculate the available power for the i^{th} component, λ^i, of the static nonlinearity. That is, we calculate the supremum in (5.3) for each of the intervals on which the nonlinearity is defined. Denote $I_1 = (-\infty, -\frac{d_{2i}}{d_{1i}})$, $I_2 = [-\frac{d_{2i}}{d_{1i}}, \frac{d_{2i}}{d_{1i}}]$, and $I_3 = (\frac{d_{2i}}{d_{1i}}, \infty)$. Then,

$$\begin{aligned}
\lambda^i_{I_1} &= \sup_{z_i \in I_1} \left\{ d_{2i}^2 - \gamma^2 z_i^2 \right\} \\
&= \left(\frac{d_{2i}}{d_{1i}} \right)^2 \left(d_{1i}^2 - \gamma^2 \right) \qquad \gamma \geq 0, \\
\lambda^i_{I_2} &= \sup_{z_i \in I_2} \left\{ \left(d_{1i}^2 - \gamma^2 \right) z_i^2 \right\} \\
&= \begin{cases} \left(\frac{d_{2i}}{d_{1i}} \right)^2 \left(d_{1i}^2 - \gamma^2 \right) & \gamma < d_{1i}, \\ 0 & \gamma \geq d_{1i}, \end{cases} \\
\lambda^i_{I_3} &= \left(\frac{d_{2i}}{d_{1i}} \right)^2 \left(d_{1i}^2 - \gamma^2 \right) \qquad \gamma \geq 0.
\end{aligned}$$

So, applying the definition of the available power (5.3) for static nonlinearities, the available power for the nonlinear block component Δ_i is given by

$$\lambda^\gamma_{2i} = \begin{cases} \left(\frac{d_{2i}}{d_{1i}} \right)^2 \left(d_{1i}^2 - \gamma^2 \right) & \gamma < d_{1i}, \\ 0 & \gamma \geq d_{1i}. \end{cases} \tag{3.32}$$

Then, the available power for the nonlinear block Δ is given by

$$\begin{aligned}
\lambda^\gamma_2 &= \sup_{z_1 \in \mathbf{R}, z_2 \in \mathbf{R}, \cdots, z_m \in \mathbf{R}} \left\{ \sum_{i=1}^{m} \left[|\alpha_i(z_i)|^2 - \gamma^2 |z_i|^2 \right] \right\} \\
&= \sum_{i=1}^{m} \left\{ \sup_{z_i \in \mathbf{R}} \left\{ |\alpha_i(z_i)|^2 - \gamma^2 |z_i|^2 \right\} \right\} \\
&= \sum_{i=1}^{m} \lambda^\gamma_{2i} \qquad\qquad (3.33)
\end{aligned}$$

Note that from (3.32), (3.33), the available power for the plant uncertainty Δ is finite for any choice of gain. Hence, it is clear that inequality (3.29) is satisfied for any choice of $\gamma_2 \geq 0$. But, presuming that the linear $\mathcal{H}_\infty$-suboptimal problem is solvable for the plant G_L, there exists a finite γ_1 which satisfies inequality (3.26). So, choosing $\gamma_2 < \frac{1}{\gamma_1}$ will ensure that the three inequalities (3.21), (3.26), and (3.27) hold simultaneously. Hence, provided that the remaining technical assumptions of Theorem 3.4.3 hold, the linear system with deadzone actuator nonlinearity will be stable in the sense that all trajectories tend to a compact set. ♦

In the above Example, we were able to conclude via Theorem 3.4.3 that the closed loop system (Δ, G_L, K_L) was stable provided that the algebraic Riccati equation (3.20) admitted a positive definite solution for a finite gain γ_1. For plants with one control, $u \in \mathbf{R}$, it is possible to find a necessary condition for existence of a positive definite solution.

Theorem 3.4.5 *Suppose that G_L is of the form (3.17), where A is unstable and Q is scalar (that is, $u \in \mathbf{R}$). Then, a necessary condition for the existence of a stabilizing positive definite solution of the algebraic Riccati equation (3.20) is that*

$$\frac{1}{\gamma_1^2} - Q^2 \quad < \quad 0 \tag{3.34}$$

Proof: Suppose there exists gain γ_1 not satisfying the inequality (3.34) such that there exists a stabilizing positive definite solution P of (3.20). Then, rewriting (3.20),

$$A'P + PA + \frac{1}{\gamma_1^2} P \left(B_2 B_2' + \left(1 - \gamma_1^2 Q^2\right) B_1 B_1' \right) P + C'C \quad = \quad 0. \tag{3.35}$$

Since $1 - \gamma_1^2 Q^2 \geq 0$, we can define the matrix $\tilde{B} = \left[\begin{array}{cc} B_2 & B_1 \sqrt{1 - \gamma_1^2 Q^2} \end{array} \right]$, so that (3.35) becomes

$$A'P + PA + \frac{1}{\gamma_1}^2 P \tilde{B} \tilde{B}' P + C'C \quad = \quad 0. \tag{3.36}$$

But, by assumption, there exists a stabilizing positive definite P satisfying (3.35), (3.36). Hence, by the Bounded Real Lemma, A is asymptotically stable, which is a contradiction. ∎

The following example demonstrates the application of Theorem 3.4.5 to a actuator deadzone problem.

Example 3.4.6 Consider the linear plant with deadzone actuator nonlinearity, G (3.13). For the deadzone (3.30), it is possible to show that the available power λ_2^γ for the corresponding plant uncertainty Δ is zero for arbitrary scalar Q provided that $\gamma \geq |Q|$. That is,

$$\gamma_2 \quad = \quad |Q| \tag{3.37}$$

is the minimal gain satisfying constraint (3.27). But, Theorem 3.4.5 requires that condition (3.34) hold. That is,

$$\gamma_1 \quad > \quad \frac{1}{|Q|}. \tag{3.38}$$

But, combining (3.37) and (3.38) violates the small gain condition (3.21). Hence, we cannot apply Corollary 3.4.2 for any value of Q. That is, we cannot guarantee that the closed loop system (Δ, G_L, K_L) is asymptotically stable. Note however, that this is expected for unstable linear plants with a zero center actuator deadzone. Asymptotic stability for such systems cannot be achieved. Note however that if the plant is stable, Theorem 3.4.5 no longer applies. That is, condition (3.34) can be violated and there still exist a positive definite solution P of (3.20) if A is stable. ♦

Example 3.4.7 Consider an unstable linear plant of the form (3.17) with $A = B_1 = B_2 = C = Q = 1$. For this system, it is easy to show that the closed loop linear system (G_L, K_L) has $\mathcal{H}_\infty$ norm $\|(G_L, K_L)\|_{\mathcal{H}_\infty} = \sqrt{2}$. Suppose that the actuator nonlinearity is a shifted unity gain deadzone (see Figure 3.7) of the form

$$\alpha(u) \quad = \quad \delta(u-2)+1, \tag{3.39}$$

where $\delta(\cdot)$ is a unity gain deadzone with deadband $[-1, 1]$.

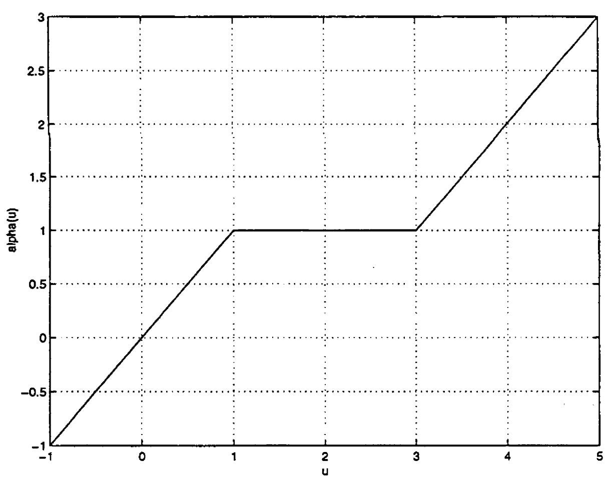


Figure 3.7: Shifted Deadzone Actuator Nonlinearity

Following steps similar to that used in the examples of Section 5.1, it can be shown that the available power for the corresponding plant uncertainty Δ is given by

$$\lambda_a^{\gamma_2} \quad = \quad \begin{cases} 4 - 9\gamma_2^2 & \gamma_2 \in [0, \frac{2}{3}), \\ 0 & \gamma_2 \in [\frac{2}{3}, \infty). \end{cases} \tag{3.40}$$

As it is possible to choose $\gamma_1 \in (\sqrt(2), \frac{10}{7})$ with $\gamma_2 = \frac{7}{10}$ such that constraints $\gamma_1\gamma_2 < 1$, clearly the constraints (3.21), (3.26), and (3.27) hold simultaneously. Furthermore, the system is detectable (trivial). Hence, by Corollary 3.4.2, the closed loop system (Δ, G_L, K_L) must be asymptotically stable in the absence of disturbances. This is demonstrated in the simulation results of Figure 3.8, where each curve represents the unperturbed trajectory for the given value of gain γ_1.

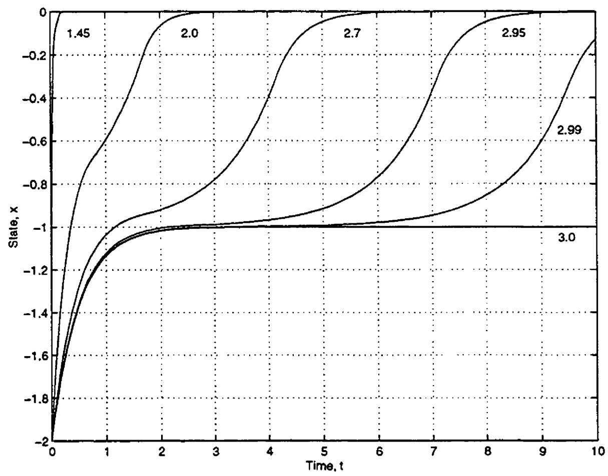


Figure 3.8: Stability for the Unstable Linear System with Shifted Deadzone Actuator Nonlinearity

It is important to note that Corollary 3.4.2 implies that the inequalities (3.21), (3.26), (3.27) (plus detectability) together form a sufficient condition for asymptotic stability of the closed loop system (Δ, G_L, K_L). Since this is not a necessary condition, we expect that for some larger values of gain γ_1 which do not satisfy the above inequalities, the system (Δ, G_L, K_L) may still exhibit asymptotic stability. This is demonstrated in Figure 3.8, from which it is evident that the system (Δ, G_L, K_L) is asymptotically stable for $\gamma_1 \in (\sqrt{2}, 3)$. If the conditions of Corollary 3.4.2 were necessary, we would expect loss of asymptotic stability for $\gamma_1 > \frac{3}{2}$. In fact, for these conditions to be necessary would require the input $\tilde{z}$ to the plant uncertainty Δ (see Figures 3.5, 3.6) to be "worst case", in that $\tilde{z}$ achieves the supremum in the available power (5.3). ♦

Finally, we consider a system for which stability in the sense of Theorem 3.4.3 cannot be guaranteed.

Example 3.4.8 Consider an unstable linear plant of the form (3.17) with $A = B_1 = B_2 = C = Q = 1$. Suppose that the actuator nonlinearity is a unity gain saturation with linear region $[-1,1]$, given by

$$\sigma(u) \;=\; u - \delta(u), \tag{3.41}$$

where $\delta(\cdot)$ is a unity gain deadzone with deadband $[-1,1]$. Applying (3.19), the plant uncertainty is clearly $-\delta(u)$. As shown in Example 5.1.1, the available power for this plant uncertainty is thus (5.12),

$$\lambda_a^{\gamma_2} \;=\; \begin{cases} \infty & \gamma_2 < 1, \\ 0 & \gamma_2 \geq 1. \end{cases}$$

So, since $\|(G_L, K_L\|_{\mathcal{H}_\infty} = \sqrt{2}$, any choice of γ_1, γ_2 satisfying the constraints (3.26), (3.29) will together violate the small gain condition (3.21). Hence, Theorem 3.4.3 cannot be applied. That is, we cannot guarantee that the system (Δ, G_L, K_L) will be stable in the sense that unperturbed trajectories tend to a compact set. Note however, that this is as expected. The plant is unstable with a saturating control. Such a plant is not globally stabilizable using the control input u. ♦

3.4.3 Power Gain Control using a Nonlinear Controller

The essential limitation of designing a linear power gain controller for a linear system with actuator nonlinearity is that the structure of the nonlinearity is ignored. This is due to the fact that the nonlinearity is recast as a plant uncertainty, and thus treated as a disturbance to the system. The linear $\mathcal{H}_\infty$ control design techniques employed to then synthesis the controller pay no heed to the actual disturbance to the system arising from plant uncertainty. Instead, the controller is designed for the "worst case" disturbance, that is, the "worst case" nonlinearity. This naturally results in a conservative controller design, and thus a degraded controller performance in the sense of minimizing closed loop oscillations.

Another approach to the problem of controller design is to consider the plant as fundamentally nonlinear. Then, by applying Theorem 3.3.6 directly, the controller obtained is guaranteed to have the best performance. However, this performance is dependent on the way in which the state and control are penalized. That is, the choice of the output z in (3.14) has a direct effect on the resulting controller performance. In particular, in this section we focus on the choice of the control penalty function $\mu(\cdot)$ (3.14).

To begin with, suppose that we choose to penalize the actuated control. That is, we choose $\mu \equiv \alpha$. Then, the PDE (3.11) becomes

$$\inf_{u \in \mathbf{R}^m} \sup_{v \in \mathbf{R}^p} \left\{ \nabla_x V(x) \cdot [Ax + B_1\alpha(u) + B_2 v] + x'C'Cx + |\alpha(u)|^2 - \gamma^2 |v|^2 \right\} = \lambda.$$

Completing the squares yields that

$$\nabla_x V(x) \cdot Ax + \frac{1}{4}\nabla_x V(x) \left(\frac{1}{\gamma^2} B_2 B_2' - B_1 B_1' \right) \nabla_x V(x)' + x'C'Cx = \lambda, \quad (3.42)$$

with the optimal control u^* and worst case disturbance v^* given by

$$\alpha(u^*) = -\frac{1}{2} B_1' \nabla_x V(x)', \quad (3.43)$$

$$v^* = \frac{1}{2\gamma^2} B_2' \nabla_x V(x)'. \quad (3.44)$$

Consider $\lambda = 0$ and $V(x) = x'Px$, where P is a positive definite solution of the algebraic Riccati equation

$$A'P + PA + P\left(\frac{1}{\gamma^2} B_2 B_2' - B_1 B_1' \right) P + C'C = 0.$$

Clearly, the pair (λ, V) is a C^1 solution pair of the PDE (3.42). Hence, a slightly weaker version of Theorem 3.3.6 (without the Lipschitz condition on solutions) would imply that the controller K^* with control policy (3.43) is optimal in th sense of Definition 3.3.5, and that the achieved closed loop available power is $\lambda_a^* = 0$. Hence, in view of the control policy (3.43), it is apparent that the optimal control policy K^* *inverts* the actuator nonlinearity $\alpha(\cdot)$, thereby reducing the closed loop to one which is linear. Hence, optimal power gain control is an example of the use of inverse models [40] in the control of linear systems with actuator nonlinearities.

Example 3.4.9 Consider the unstable scalar nonlinear plant

$$G : \begin{cases} \dot{x} = x + \delta(u) + v, \\ z = \begin{bmatrix} x \\ \delta(u) \end{bmatrix}, \end{cases}$$

where $\delta(\cdot)$ is a unity gain deadzone with deadband $[-1, 1]$. The aim is to find numerically the optimal power gain controller with gain ≤ 2. To apply the mixed policy / value space methods of Chapter 4, define the state space coordinate grid $(R)^{\delta X}$ with spacing $\delta_X = 0.01$. Restrict the coordinate grid to $G_X = (\mathbf{R})^{\delta_X} \cap [-1, 1]$. Similarly, define the restricted coordinate grids for the controls and disturbances as $G_U = (\mathbf{R})^{0.01} \cap [-4, 4]$ and $G_V = (\mathbf{R})^{0.02} \cap [-1, 1]$. Choose the number of value space iterations per policy space iteration to be $N = 100$, with a total of 20 policy space iterations to be performed.

Running the finite difference scheme (a few minutes on a standard UNIX platform) yields a closed loop available power approximation equal to 0 and corresponding optimal $\mathcal{FP}$-gain controller as shown in Figure 3.9 (note that the dashed line represents the $\mathcal{H}_\infty$ controller for $\delta(u) = u$).

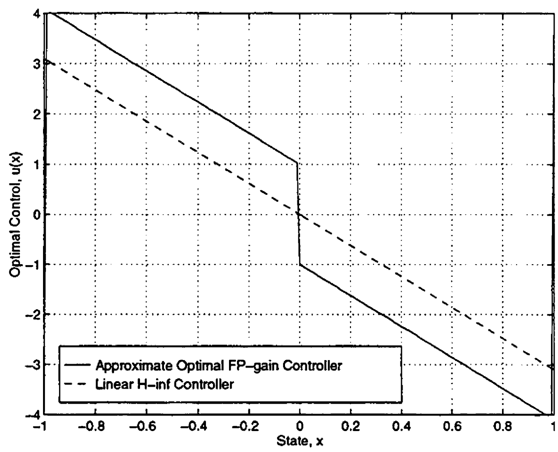


Figure 3.9: Optimal Power Gain Controller for Gain ≤ 2 (Example 3.4.9)

The actuated control $\tilde{u}(x)$ is given by the composition $\delta \circ u(x)$ and is illustrated in Figure 3.10. Hence, in minimizing the available power, the optimization procedure naturally seeks the inverse of the nonlinearity, thereby yielding an asymptotically stable closed loop. Note that as expected, the value function is quadratic, as shown in Figure 3.11. ♦

Example 3.4.10 Consider the same system in Example 3.4.9, but with $\alpha(u) = 2.5u(u^2 - 1)$. In this case, more than one inverse for the nonlinearity exists. For a gain of $\gamma = 2$, choose the restricted coordinate grids $G_X = (\mathbf{R})^{0.01} \cap [-1.5, 1.5]$, $G_U = (\mathbf{R})^{0.01} \cap [-1.5, 1.5]$, and $G_V = (\mathbf{R})^{0.05} \cap [-1.0, 1.0]$. Using the same number of policy space and value space iterations as in Example 3.4.9, the available power approximation is again 0, with the corresponding optimal $\mathcal{FP}$-gain controller as shown in Figure 3.12. The composition $\alpha \circ u(x)$ (the actuated control) again yields the linear $\mathcal{H}_\infty$ controller for the linear system with $\alpha(u) = u$, as in Figure 3.10. Note that in the region where more than one inverse for the nonlinearity exists, the inverse chosen is arbitrary, depending on the implementation of the finite difference approximation code. ♦

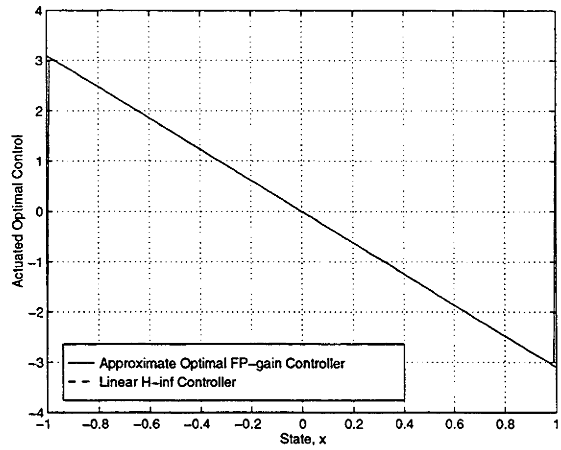


Figure 3.10: Actuated Optimal $\mathcal{FP}$-gain Control $\tilde{u}(x)$ (Example 3.4.9)

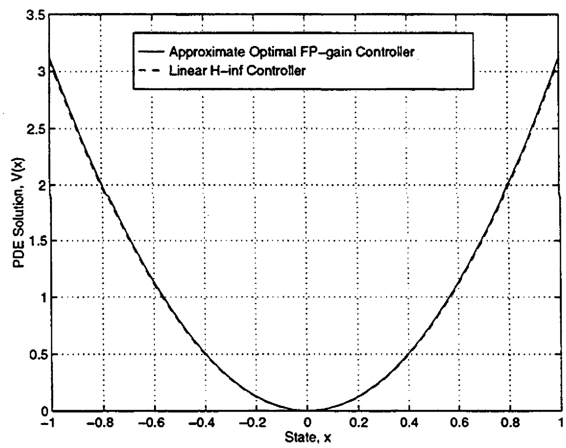


Figure 3.11: Value Function Approximation $V(x)$ (Example 3.4.9)

The examples presented demonstrate that if the nonlinearity $\alpha(\cdot)$ is invertible, then the optimal power gain controller from Theorem 3.3.6 yields zero available power in the closed loop. That is, the optimization is effectively over the class of controllers

$$K \ : \ u = \left(\alpha^{-1}\right)(Lx),$$

where $L \in \mathbf{R}^{m \times n}$ is the linear $\mathcal{H}_\infty$ controller for the plant with the nonlinearity set to identity (see [40]). With the closed loop then behaving linearly, the available power must be zero.

Finally, as an alternative to penalizing the actuated control, suppose that the control itself is now penalized. That is, choose $\mu(u) = u$ in (3.14). Then, returning once again

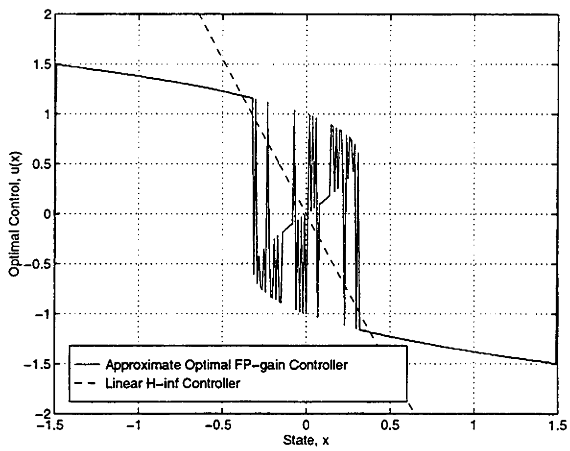


Figure 3.12: Optimal $\mathcal{FP}$-gain Controller $u(x)$ (Example 3.4.10)

to the PDE (3.11),

$$\inf_{u\in\mathbf{R}^m}\sup_{v\in\mathbf{R}^p}\left\{\nabla_x V(x)\cdot[Ax+B_1\alpha(u)+B_2v]+x'C'Cx+|u|^2-\gamma^2|v|^2\right\} \;=\; \lambda.$$

Proceeding as with (3.42), the next step is to complete the squares. However, completion of squares for the infimum over u is no longer possible since the PDE is no longer quadratic in u. Instead, the PDE to solve is

$$\inf_{u\in\mathbf{R}^m}\left\{\nabla_x V(x)\cdot Ax+\frac{1}{4\gamma^2}\nabla_x V(x)B_2B_2'\nabla_x V(x)'+x'C'Cx+\nabla_x V(x)B_1\alpha(u)+|u|^2\right\} = \lambda. \tag{3.45}$$

Assuming that the actuator nonlinearity is differentiable, the optimal control u^* and worst case disturbance v^* are now given by

$$u^* = -\frac{1}{2}\left.\frac{d\alpha}{du}\right|_u^* B_1'\nabla_x V(x)', \tag{3.46}$$

$$v^* = \frac{1}{2\gamma^2}B_2'\nabla_x V(x)'. \tag{3.47}$$

Note that (3.46) is now a recursive prescription for the optimal control u^*. Consequently, explicit computation of the optimal power gain controller is much more difficult than in the case of penalizing the actuated control.

Example 3.4.11 Consider the unstable scalar nonlinear plant

$$G : \begin{cases} \dot{x} = x+\delta(u)+v, \\ z = \begin{bmatrix} x \\ u \end{bmatrix}, \end{cases}$$

where $\delta(\cdot)$ is a unity gain deadzone with deadband $[-1,1]$. Applying Theorem 3.3.6 for a gain $\gamma = 2$ yields an available power of $\lambda_a = \frac{4}{3}$, and a storage function as shown in Figure 3.13. The optimal power gain controller with gain ≤ 2 can be shown to be

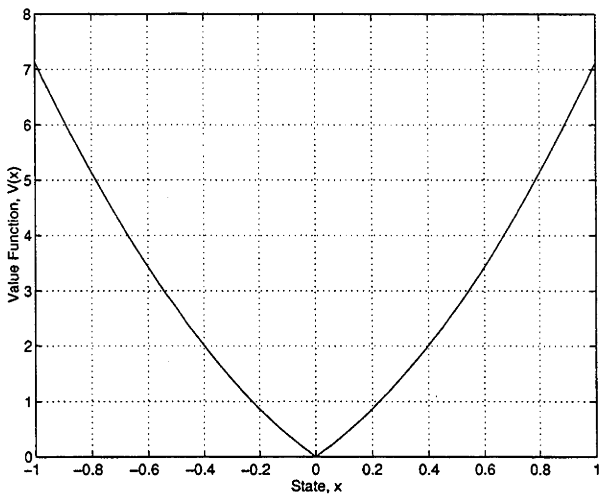


Figure 3.13: Value Function $V(x)$ (Example 3.4.11)

given by

$$u(x) = \begin{cases} -\frac{4}{3}(x-1) + \frac{4}{3}\sqrt{\frac{7}{4}x^2 - 2x} & x < 0, \\ 0 & x = 0, \\ -\frac{4}{3}(x+1) - \frac{4}{3}\sqrt{\frac{7}{4}x^2 + 2x} & x > 0. \end{cases}$$

The closed loop behaviour of the system is illustrated in Figure 3.14. Note that if we compute an approximation to the available power using (2.84) and the simulation results of Figure 3.14, we again find that $\lambda_a = \frac{4}{3}$. ♦

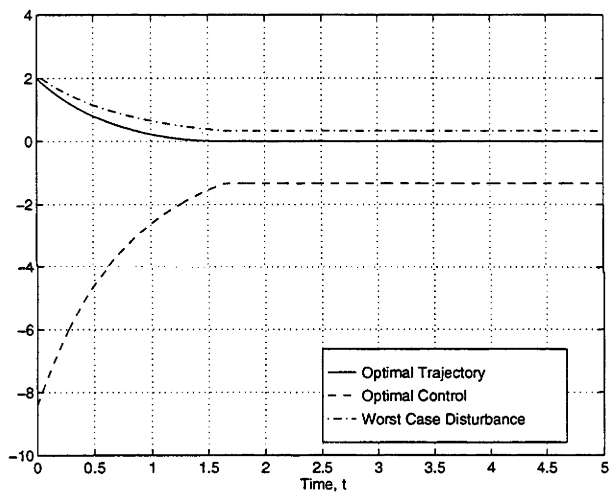


Figure 3.14: Optimal Trajectory, Optimal Control, and Worst Case Disturbance (Example 3.4.11)

Chapter 4

Numerical Methods: Finite Differences

4.1 Introduction

One of the main aims of Chapter 2 was to analyse a class of nonlinear systems with a view to linking the input / output property of power gain to a particular form of internal stability. In establishing this link, it was useful to generalize the notion of dissipativity [42, 21, 22] to include systems with power gain. This generalization facilitated the expression of the power gain property as a differential inequality, thereby establishing a verification result for the power gain property for nonlinear systems. As a continuation of this verification result, it was found to be possible to to propose a number of explicit candidate power bias / storage function pairs for which the dissipation inequality, and hence the differential inequality, were shown to hold.

Since the candidate power bias / storage function pairs are inherently variational in form, explicit computation of these quantities is often difficult. Hence, in this Chapter it is our goal to be able to compute numerical approximations for these candidate power bias / storage function pairs directly from their respective definitions. Note that much of the wisdom in deriving the approximations presented is attributable to the treatment of similar ergodic stochastic control problem approximations presented in [29].

The importance of the approximations presented in this chapter will become fully apparent in Chapter 5, where many of the methods are used to verify numerically the results of Chapter 2. Note that although similar approximations may be developed for the control problems of Chapter 3, this chapter focuses only on the analysis problems of Chapter 2.

Finally, a important topic in future work will be the proof of convergence of the methods presented. Although many results exist in the stochastic case [29], treatment of the deterministic case remains largely outstanding.

4.2 Fundamental Finite Horizon Computations

Referring again to Chapter 2, the candidate power bias / storage function pairs of interest are (λ_a, V_a), (λ_a, V_b), and $(\lambda_a, V_{br}^f(\xi, x))$, where λ_a (2.84) is the available power, $V_a(x)$ (2.149) is the super available storage, $V_b(x)$ (2.169) is the infinite horizon available storage, and $V_{br}^f(\xi, x)$ (2.199) is the fixed initial state required supply. As was noted in Chapter 2, these quantities may be expressed in terms of the finite horizon value functions $V(x,T)$ (2.33) and $V_r^f(\xi, x, T)$ (2.194). Recalling (2.85), (2.150), (2.170), and (2.200),

$$\begin{aligned}
\lambda_a &= \limsup_{T\to\infty} \left\{ \frac{V(x,T)}{T} \right\}, && (4.1)\\
V_a(x) &= \sup_{T\geq 0} \{V(x,T) - \lambda_a T\}, && (4.2)\\
V_b(x) &= \limsup_{T\to\infty} \{V(x,T) - \lambda_a T\}, && (4.3)\\
V_{br}^f(\xi, x) &= \liminf_{T\to\infty} \left\{ V_r^f(\xi, x, T) + \lambda_a T \right\}, && (4.4)
\end{aligned}$$

where

$$\begin{aligned}
V(x,T) &= \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[c(x(s)) - \gamma^2 |v(s)|^2\right] \, ds : x(0) = x \right\}, && (4.5)\\
V_r^f(\xi, x, T) &= \inf_{v\in\mathcal{L}_2[-T,0]} \left\{ \int_{-T}^0 \left[\gamma^2 |v(s)|^2 - c(x(s))\right] \, ds : x(-T) = \xi, x(0) = x \right\}. && (4.6)
\end{aligned}$$

It is clear from these equations that the computation of the finite horizon value functions $V(x,T)$ (4.5) and $V_r^f(\xi, x, T)$ (4.6) is fundamental to the computation of the candidate power bias / storage function pairs from their respective definitions.

In order that the finite horizon value function $V(x,T)$ satisfy the nonstationary PDE (2.50), it is necessary to make the following assumption:

(A18) The finite horizon value function $V(x,T)$ is continuous.

See [32] for a proof of assumption (A18).

4.3 A Centered Method for (λ_a, V_b)

Assuming that (A18) holds, Theorem 2.5.10 implies that the finite horizon value function $V(x,T)$ is a continuous viscosity solution of the nonstationary PDE (2.50). That is,

$$\frac{\partial V}{\partial T}(x,T) = H(x, \nabla_x V(x,T)), \quad V(x,0) = 0. \tag{4.7}$$

In this section, we apply the standard finite difference approximation [29] to the nonstationary PDE (4.7) to obtain an iterative approximation scheme for $V(x,T)$ (2.33). By applying definitions (4.1) and (4.3), this scheme can be adapted to compute the pair (λ_a, V_b).

From Section 5.3 of [29], the gradient of V, $\nabla_x V$, may be approximated by a one sided difference. The i^{th} component of this finite difference is

$$\nabla_{x,i}^{\delta_X} V(x,T) = \begin{cases} \frac{V(x,T)-V(x-\delta_X e_i,T)}{\delta_X} & g_i(x,v) < 0, \\ \frac{V(x+\delta_X e_i,T)-V(x,T)}{\delta_X} & g_i(x,v) \geq 0, \end{cases} \tag{4.8}$$

where $\delta_X > 0$ is the spacing, e_i is the i^{th} basis vector in $\mathbf{R}^n$, and $g_i(x,v) = a_i(x) + \sum_{j=1}^{p} \{b_{ij}(x)v_j\}$ (the i^{th} component of the drift). Note that the term $\nabla_{x,i}^{\delta_X} V(x)$ is used to denote the difference approximation to the gradient component $\nabla_{x,i} V(x)$.

In order to reduce the case by case definition of (4.8) to a single expression, define the i^{th} component of the indicator function χ to be

$$\chi_i(x,v) = \begin{cases} -1 & g_i(x,v) < 0, \\ 0 & g_i(x,v) = 0, \\ 1 & g_i(x,v) > 0. \end{cases} \tag{4.9}$$

Then, the finite difference approximation to the i^{th} component of the derivative with respect to s of $V(x(s),T)$ along the trajectory $x(s)$ is given by

$$\begin{aligned} \nabla_{x,i}^{\delta_X} V(x,T) \cdot g_i(x,v) &= \begin{cases} \left[\frac{V(x,T)-V(x-\delta_X e_i,T)}{\delta_X}\right] g_i(x,v) & g_i(x,v) < 0, \\ 0 & g_i(x,v) = 0, \\ \left[\frac{V(x+\delta_X e_i,T)-V(x,T)}{\delta_X}\right] g_i(x,v) & g_i(x,v) > 0, \end{cases} \\ &= \left[\frac{V(x+\chi_i(x,v)\delta_X e_i,T) - V(x,T)}{\delta_X}\right] g_i(x,v)\chi_i(x,v). \end{aligned} \tag{4.10}$$

A first step in computing the finite difference approximation of $V(x,T)$ is to consider a restriction of the state space and disturbance space to compact sets on respective coordinate grids,

$$G_X = \{x \in \mathbf{R}^n : |x| \le K_X\} \cap (\mathbf{R}^n)^{\delta_X}, \tag{4.11}$$

$$G_V = \{v \in \mathbf{R}^p : |x| \le K_V\} \cap (\mathbf{R}^p)^{\delta_V}, \tag{4.12}$$

where K_X, K_V are finite positive constants, δ_V is the disturbance grid spacing, and $(\mathbf{R}^{n,p})^{\delta_{X,V}}$ denotes the coordinate grids in $\mathbf{R}^n$, $\mathbf{R}^p$ with spacing δ_X, δ_V respectively. With the disturbance coordinate grid imposed, the maximum speed of the dynamics for each state x in the restricted space G_X is

$$m(x) = \max_{v \in G_V} |g(x,v)|_1, \tag{4.13}$$

where $|\cdot|_1$ represents the 1-norm on $\mathbf{R}^n$. From this, we define the interpolation time

$$\delta_T(x) = \frac{\delta_X}{m(x)}. \tag{4.14}$$

Given the simplex $N^{\delta_X}(x)$,

$$N^{\delta_X}(x) = \{x \pm \delta_X e_i : i = 1, \cdots, n\} \cup \{x\}, \tag{4.15}$$

the interpolation time can be interpreted as representing the minimum time required for the state to move from x to any other state in the simplex $N^{\delta_X}(x)$ (see Figure 4.1). For this reason, the interpolation time $\delta_T(x)$ can be used as the basic unit of time for

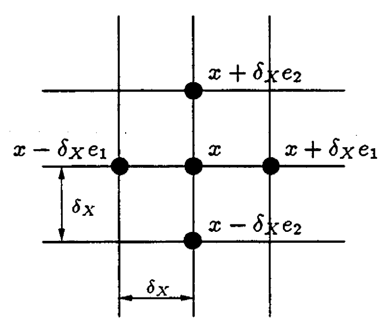


Figure 4.1: Simplex $N^{\delta_X}(x)$ in G_X for $\mathbf{R}^2$

state $x \in G_X$ in the iterative scheme. However, for simplicity, the more convervative minimal interpolation time δ_T may be used, where

$$\begin{aligned} \delta_T &= \min_{x \in G_X} \{\delta_T(x)\} \\ &= \frac{\delta_X}{m}, \end{aligned} \tag{4.16}$$

and $m = \max_{x \in G_X} \{m(x)\}$. So, applying this definition of minimal interpolation time

δ_T (4.16) to the i^{th} component of the finite difference approximation (4.10),

$$\nabla^{\delta_X}_{x,i} V(x,T) \cdot g_i(x,v) \;=\; \left[\frac{V(x+\chi_i(x,v)\delta_X e_i.T) - V(x,T)}{\delta_T}\right]\left(\frac{|g_i(x,v)|}{m}\right).$$

Hence, the finite difference approximation of $\nabla_x V(x,T)$ is

$$\begin{aligned}
&\nabla^{\delta_X}_{x} V(x,T) \cdot g(x,v) \\
&= \sum_{i=1}^{n} \left[\nabla^{\delta_X}_{x,i} V(x,T) \cdot g_i(x,v)\right] \\
&= \frac{1}{\delta_T}\sum_{i=1}^{n}\left[\left[V(x+\chi_i(x,v)\delta_X e_i, T) - V(x,T)\right]\left(\frac{|g_i(x,v)|}{m}\right)\right] \\
&= \frac{1}{\delta_T}\left\{-V(x,T) + V(x,T)\left[1 - \sum_{i=1}^{n}\frac{|g_i(x,v)|}{m}\right] + \right. \\
&\qquad\qquad \left. \sum_{i=1}^{n}\left[V(x+\chi_i(x,v)\delta_X e_i, T)\frac{|g_i(x,v)|}{m}\right]\right\} \qquad (4.17)
\end{aligned}$$

Define the positive and negative envelopes of the dynamics $g(x,v)$ to be

$$g_i^+(x,v) \;=\; \begin{cases} 0 & g_i(x,v) < 0, \\ g_i(x,v) & g_i(x,v) \geq 0, \end{cases}$$

$$g_i^-(x,v) \;=\; \begin{cases} -g_i(x,v) & g_i(x,v) < 0, \\ 0 & g_i(x,v) \geq 0. \end{cases}$$

Clearly $g_i^+(x,v) + g_i^-(x,v) = |g_i(x,v)|$ and $g_i^+(x,v) - g_i^-(x,v) = g_i(x,v)$. Furthermore, $\chi_i(x,v) = 1$ implies $g_i^+(x,v) > 0$, and $\chi_i(x,v) = -1$ implies $g_i^-(x,v) > 0$. So, (4.17) becomes

$$\begin{aligned}
\nabla^{\delta_X}_{x} V(x,T) \cdot g(x,v) \;&=\; \frac{1}{\delta_T}\left\{-V(x,T) + V(x,T)\left[1 - \frac{|g_i(x,v)|_1}{m}\right] + \right. \\
&\qquad \left. \sum_{i=1}^{n}\left[V(x+\delta_X e_i, T)\frac{g_i^+(x,v)}{m} + V(x-\delta_X e_i, T)\frac{g_i^-(x,v)}{m}\right]\right\} \\
&=\; \frac{1}{\delta_T}\left\{-V(x,T) + \sum_{z \in N^{\delta_X}(x)} \left[V(z,T)p(x,z|v)\right]\right\}, \qquad (4.18)
\end{aligned}$$

where

$$p(x,z|v) \;=\; \begin{cases} 1 - \frac{|g(x,v)|_1}{m} & z = x, \\ \frac{g_i^{\pm}(x,v)}{m} & z = x \pm \delta_X e_i, \\ 0 & \text{elsewhere.} \end{cases} \qquad (4.19)$$

Note that from the derivation, $\sum_{z \in N^{\delta_X}(x)} p(x,z|v) = 1$ for any $v \in G_V$. That is, $p(x,z|v)$ may be regarded as a transition probability from state x to state z, given the disturbance v. Although this interpretation is not required for the implementation of a finite difference method, it is useful for proving convergence of the scheme. See [29].

Equations (4.18), (4.19) provide a concise way of writing the finite difference approximation for $\nabla_x V(x,T)$. However, referring again to the nonstationary PDE (4.7), a finite difference approximation for the time derivative $\frac{\partial V}{\partial T}(x,T)$ is also required. But, for this approximation, we can apply the interpretation of the minimal interpolation time δ_T as the minimum transition time between states x and $z \in N^{\delta X}(x)$, for any $x \in G_X$. That is,

$$\frac{\partial V}{\partial T}(x,T) \approx \frac{V(x,T+\delta_T) - V(x,T)}{\delta_T}. \tag{4.20}$$

So, substituting the finite difference approximations (4.18) and (4.20) into the nonstationary PDE (4.7), and limiting the disturbance space to G_V,

$$\frac{V(x,T+\delta_T) - V(x,T)}{\delta_T} = \max_{v \in G_V} \left\{ \frac{1}{\delta_T} \left[-V(x,T) + \sum_{z \in N^{\delta X}(x)} [V(z,T)p(x,z|v)] \right] + c(x) - \gamma^2 |v|^2 \right\}.$$

Multiplying through by δ_T and cancelling the $V(x,T)$ term yields the recursion

$$V^\delta(x,T+\delta_T) = \max_{v \in G_V} \left\{ \sum_{z \in N^{\delta X}(x)} \left[V^\delta(z,T)p(x,z|v) \right] + \delta_T \left[c(x) - \gamma^2 |v|^2 \right] \right\}, \tag{4.21}$$

where the δ superscript denotes the use of the finite difference approximation. Since the minimal interpolation time δ_T is the time increment in the recursion (4.21), it is natural to define the k^{th} iteration as

$$V_k^\delta(x) = V^\delta(x, k\delta_T), \tag{4.22}$$

where $k = \left\lfloor \frac{T}{\delta_T} \right\rfloor$, and the δ superscript serves as a reminder that V_k^δ is an approximation. Hence, applying (4.22) to (4.21) yields an iterative scheme for computing the finite difference approximation for $V(x,T)$ (4.5),

$$V_k^\delta(x) = \max_{v \in G_V} \left\{ \sum_{z \in N^{\delta X}(x)} \left[V_k^\delta(z)p(x,z|v) \right] + \delta_T \left[c(x) - \gamma^2 |v|^2 \right] \right\}. \tag{4.23}$$

From the definition of the available power λ_a (4.1), it is clear that for systems with nonzero available power, the finite horizon value function $V(x,T)$ grows asymptotically linearly with T. This implies that the approximation V_k^δ grows asymptotically linearly with k, and hence, in an unbounded fashion. From a numerical standpoint, this type of "uncentered" scheme is unsatisfactory. However, since the aim is to compute the pair (λ_a, V_b) rather than $V(x, T)$ itself, this problem is avoided. As will become clear, the modification of the scheme (4.23) required to compute the pair (λ, V_b) results in a centering step. This prevents the unbounded growth of numerical quantities in any implemented algorithm for the scheme.

It is important at this point to stress that the iterative scheme (4.23) utilizes a fixed time increment given by the minimal interpolation time δ_T (4.16). Hence, the sequence of time horizons used in computing any approximation for the pair (λ_a, V_b) is fixed. For this reason, it is imperative that the limits exist in both the definition of the available power λ_a (4.1), and the infinite horizon available storage $V_b(x)$ (4.3). Under the required assumptions, Theorem 2.6.6 implies that this is the case for the available power λ_a (4.1). However, it remains to be shown what assumptions are required for the limit to exist for the infinite horizon available storage V_b (4.3).

(A19) The limit in the definition of the infinite horizon available storage $V_b(x)$ (4.3) exists. That is,

$$V_b(x) \;=\; \lim_{T\to\infty} \{V(x,T) - \lambda_a T\}\,.$$

With assumption (A19) enacted, it is clear from (4.3) that

$$V_b(x) - V_b(x_0) \;=\; \lim_{T\to\infty} \{V(x,T) - V(x_0,T)\}$$

for any reference state $x_0 \in \mathbf{R}^n$. So, for sufficiently large iteration k, this may be approximated by

$$\tilde{V}^{\delta}_{b,k}(x) \;:=\; V^{\delta}_{b,k}(x) - V^{\delta}_{b,k}(x_0) \;=\; V^{\delta}_{k}(x) - V^{\delta}_{k}(x_0), \qquad (4.24)$$

where $V^{\delta}_{b,k}(x)$ denotes the k^{th} approximation of $V_b(x)$, and $V^{\delta}_{k}(x)$ denotes the k^{th} approximation of $V(x, k\delta_T)$, (4.22). Since (4.24) involves a difference in the approximations for $V(x,T)$, any asymptotic linear growth in the approximation $V^{\delta}_{k}(x)$ must be absent in the approximation $V^{\delta}_{b,k}(x) - V^{\delta}_{b,k}(x_0)$. So, (4.24) is precisely the centering step for computing the approximation for $V^{\delta}_{b,k}(x) - V^{\delta}_{b,k}(x_0)$. The immediate disadvantage of this centering step is that $V_b(x)$ can only be approximated to within an unknown additive constant, $V_b(x_0)$. Any centered scheme will produce an approximation which is zero at the reference state, resulting in the approximation differing from the actual function by a constant (dependent on x_0). Fortunately however, this is not a problem with the bisection schemes (Section 4.6), so that this constant of normalization can be computed.

In any case, rewriting (4.24),

$$V_k^\delta(x) \;=\; \tilde{V}_{b,k}^\delta(x) + V_k^\delta(x_0). \tag{4.25}$$

Substituting (4.25) into the RHS of the iterative scheme (4.23),

$$\begin{aligned}
V_{k+1}^\delta(x) &= \max_{v\in G_V}\left\{ \sum_{z\in N^\delta X(x)} \left[\tilde{V}_{b,k}^\delta(z)p(x,z|v)\right] + \sum_{z\in N^\delta X(x)} \left[V_k^\delta(x_0)p(x,z|v)\right] + \right. \\
&\qquad\qquad \left. \delta_T\left[c(x)-\gamma^2|v|^2\right]\right\} \\
&= \max_{v\in G_V}\left\{ \sum_{z\in N^\delta X(x)} \left[\tilde{V}_{b,k}^\delta(z)p(x,z|v)\right] + V_k^\delta(x_0) + \delta_T\left[c(x)-\gamma^2|v|^2\right]\right\} \qquad (4.26)
\end{aligned}$$

since $\sum_{z\in N^\delta X(x)} p(x,z|v) = 1$ for all $v \in G_V$. Defining

$$\tilde{V}_{k+1}^\delta(x) \;=\; V_{k+1}^\delta(x) - V_k^\delta(x_0), \tag{4.27}$$

the iteration (4.26) becomes

$$\tilde{V}_{k+1}^\delta(x) \;=\; \max_{v\in G_V}\left\{ \sum_{z\in N^\delta X(x)} \left[\tilde{V}_{b,k}^\delta(z)p(x,z|v)\right] + \delta_T\left[c(x)-\gamma^2|v|^2\right]\right\}, \tag{4.28}$$

which is essentially an incomplete centered centered version of (4.23). To complete the centering, consider again the approximation (4.25). For the $k+1^{th}$ iteration,

$$\begin{aligned}
\tilde{V}_{b,k+1}^\delta(x) &= V_{k+1}^\delta(x) - V_{k+1}^\delta(x_0) \\
&= V_{k+1}^\delta(x) - V_k^\delta(x_0) - \left(V_{k+1}^\delta(x_0) - V_k^\delta(x_0)\right) \\
&= \tilde{V}_{k+1}^\delta(x) - \tilde{V}_{k+1}^\delta(x_0), \qquad (4.29)
\end{aligned}$$

where $\tilde{V}_k^\delta$ is given by (4.27). Together, equations (4.28) and (4.29) form the basic centered finite difference scheme for approximating the function $V_b(x) - V_b(x_0)$ (via $\tilde{V}_{b,k}^\delta$).

To compute an approximation for the available power λ_a (4.1), note that $V_{k=0}^\delta(x) = 0$ for all $x \in G_X$, from the initial data for PDE (4.7). Then, $V_k^\delta(x_0)$ may be written as the telescoping series

$$\begin{aligned}
V_k^\delta(x_0) &= \left[V_k^\delta(x_0) - V_{k-1}^\delta(x_0)\right] + \left[V_{k-1}^\delta(x_0) - V_{k-2}^\delta(x_0)\right] + \cdots + \left[V_1^\delta(x_0) - V_0^\delta(x_0)\right] \\
&= \tilde{V}_k^\delta(x_0) + \sum_{i=1}^{k-1} \tilde{V}_i^\delta(x_0). \qquad (4.30)
\end{aligned}$$

Defining $R_k^\delta = V_k^\delta(x_0)$ and applying (4.30),

$$R_{k+1}^\delta \;=\; R_k^\delta + \tilde{V}_{k+1}^\delta(x_0) \tag{4.31}$$

But, from (4.1) and (4.22), an asymptotic approximation for the available power λ_a is

$$\begin{aligned} \lambda^\delta_{a,k} &= \frac{V^\delta_k(x_0)}{k\delta_T} \\ &= \frac{R^\delta_k}{k\delta_T}. \end{aligned} \tag{4.32}$$

Equations (4.28), (4.29), (4.31), and (4.32) together form the complete centered finite difference scheme for computing an approximation to the pair $(\lambda_a, V_b(x)-V_b(x_0))$. That is,

$$\tilde{V}^\delta_{k+1}(x) = \max_{v\in G_V}\left\{ \sum_{z\in N^\delta X(x)} \left[\tilde{V}^\delta_{b,k}(z)p(x,z|v)\right] + \delta_T\left[c(x)-\gamma^2|v|^2\right]\right\}, \tag{4.33}$$

$$\tilde{V}^\delta_{b,k+1}(x) = \tilde{V}^\delta_{k+1}(x) - \tilde{V}^\delta_{k+1}(x_0), \tag{4.34}$$

$$R^\delta_{k+1} = R^\delta_k + \tilde{V}^\delta_{k+1}(x_0), \tag{4.35}$$

$$\lambda^\delta_{a,k} = \frac{R^\delta_k}{k\delta_T}, \tag{4.36}$$

where $(\lambda^\delta_{a,k}, \tilde{V}^\delta_{b,k}(x))$ is the k^{th} approximation to the pair $(\lambda_a, V_b(x) - V_b(x_0))$. The initialization for the scheme follows from (4.25), (4.22), and the fact that $V(x,0) = 0$ for all $x \in \mathbf{R}^n$. That is,

$$\tilde{V}^\delta_{b,k=0}(x) = 0, \quad \text{for all } x \in G_X, \tag{4.37}$$

$$R^\delta_{k=0} = 0. \tag{4.38}$$

4.4 Accelerating Convergence of the Value Function

In deriving the centered finite difference scheme (4.33) - (4.36), the minimal interpolation time δ_T (4.16) was used as the basic unit of time in the finite difference approximation. However, this can be overly conservative given that the maximum speed of the dynamics $m(x)$ (4.13) can vary widely over the restricted state space G_X. Consequently, to improve the covergence rate of the centered scheme, it is natural to consider the basic unit of time to be state dependent [29]. That is, the state dependent interpolation time $\delta_T(x)$ can be used in place of the minimal interpolation time δ_T as the basic unit of time. An immediate consequence of this modification is that the rate of convergence of the finite difference approximation will be state dependent. This is clear from the notation (4.22),

$$V^\delta_k(x) = V^\delta(x, k\delta_T(x)). \tag{4.39}$$

For states $x,\ z \in G_X$, $\delta_T(x) \neq \delta_T(z)$, the k^{th} iteration of a finite difference scheme implies an approximation of the finite horizon value function for different time horizons, namely $k\delta_T(x)$ and $k\delta_T(z)$. This results in the introduction of a "normalization" term in the finite difference scheme, which has the effect of compensating for the differing time horizons.

In order to follow the same steps as in Section 4.3, the following assumption is required:

(A20) $V_k^\delta(x)$ is asymptotically linear in k.

In view of the definition of λ_a (4.1), assumption (A20) is acceptable since $V_k^\delta(x)$ is an approximation for the finite horizon value function $V(x, k\delta_T(x))$.

With assumptions (A18), (A19), and (A20), consider again the finite difference approximation (4.21) of the nonstationary PDE (4.7), with the state dependent interpolation time $\delta_T(x)$ (4.14) replacing the minimal interpolation time δ_T (4.16),

$$V^\delta(x, T+\delta_T(x)) = \max_{v \in G_V} \left\{ \sum_{z \in N^\delta X(x)} \left[V^\delta(z,T) p(x,z|v) \right] + \delta_T(x) \left[c(x) - \gamma^2 |v|^2 \right] \right\}. \tag{4.40}$$

For sufficiently large T, assumption (A19) implies that

$$\tilde{V}_b^\delta(x,T) := V_b^\delta(x,T) - V_b^\delta(x_0,T) = V^\delta(x,T) - V^\delta(x_0,T). \tag{4.41}$$

Hence, combining (4.40) and (4.41),

$$V^\delta(x, T+\delta_T(x)) - V^\delta(x_0,T) = \max_{v \in G_V} \left\{ \sum_{z \in N^\delta X(x)} \left[\tilde{V}_b^\delta(z,T) p(x,z|v) \right] + \delta_T(x) \left[c(x) - \gamma^2 |v|^2 \right] \right\}. \tag{4.42}$$

In order to apply the notation (4.39), it is important to recognize that with a state dependent interpolation time, the k^{th} iteration implies different horizons for different states. Hence, it is more useful to write (4.40) in terms of the iteration k and the interpolation time $\delta_T(x)$,

$$V^\delta(x, (k+1)\delta_T(x)) - V^\delta(x_0, k\delta_T(x)) = \max_{v \in G_V} \left\{ \sum_{z \in N^\delta X(x)} \left[\tilde{V}_b^\delta(z, k\delta_T(x)) p(x,z|v) \right] + \delta_T(x) \left[c(x) - \gamma^2 |v|^2 \right] \right\}. \tag{4.43}$$

Applying assumption (A20),

$$V^{\delta}(x_0, k\delta_T(x)) = V^{\delta}(x_0, k\delta_T(x_0)) \left(\frac{\delta_T(x)}{\delta_T(x_0)}\right) \tag{4.44}$$

for large k. Assumption (A19) implies that

$$\tilde{V}_b^{\delta}(z, k\delta_T(x)) = \tilde{V}_b^{\delta}(z, k\delta_T(z)) \tag{4.45}$$

for large k. So, combining (4.43), (4.44), (4.45), and notation (4.39),

$$\tilde{V}_{k+1}^{\delta}(x) = \max_{v\in G_V}\left\{\sum_{z\in N^{\delta}X(x)} \left[\tilde{V}_{b,k}^{\delta}(z)p(x,z|v)\right] + \delta_T(x)\left[c(x) - \gamma^2|v|^2\right]\right\}.$$

where

$$\tilde{V}_{k+1}^{\delta}(x) = V_{k+1}^{\delta}(x) - V_{k+1}^{\delta}(x_0)\left(\frac{\delta_T(x)}{\delta_T(x_0)}\right).$$

Following the same procedure as in Section 4.3 then yields the following revised centered finite difference scheme:

$$\tilde{V}_{k+1}^{\delta}(x) = \max_{v\in G_V}\left\{\sum_{z\in N^{\delta}X(x)} \left[\tilde{V}_{b,k}^{\delta}(z)p(x,z|v)\right] + \delta_T(x)\left[c(x) - \gamma^2|v|^2\right]\right\}, \tag{4.46}$$

$$\tilde{V}_{b,k+1}^{\delta}(x) = \tilde{V}_{k+1}^{\delta}(x) - \tilde{V}_{k+1}^{\delta}(x_0)\left(\frac{\delta_T(x)}{\delta_T(x_0)}\right), \tag{4.47}$$

$$R_{k+1}^{\delta} = R_k^{\delta} + \tilde{V}_{k+1}^{\delta}(x_0), \tag{4.48}$$

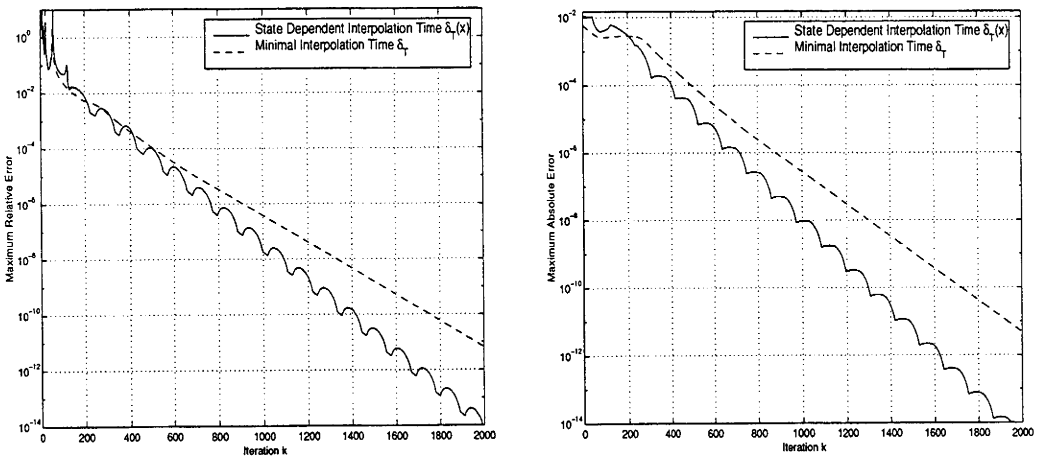
$$\lambda_{a,k}^{\delta} = \frac{R_k^{\delta}}{k\delta_T(x_0)}, \tag{4.49}$$

where $(\lambda_{a,k}^{\delta}, \tilde{V}_{b,k}^{\delta}(x))$ is the k^{th} approximation to the pair $(\lambda_a, V_b(x) - V_b(x_0))$, and

$$p(x,z|v) = \begin{cases} 1 - \frac{|g(x,v)|_1}{m(x)} & z = x, \\ \frac{g_i^{\pm}(x,v)}{m(x)} & z = x \pm \delta_X e_i, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.50}$$

The initialization for the scheme is given by (4.37) and (4.38). In comparing scheme (4.46) - (4.49) with (4.33) - (4.36), it is clear that main modification is the time horizon normalizing term $\left(\frac{\delta_T(x)}{\delta_T(x_0)}\right)$ in (4.47).

In order to compare the performance of schemes (4.46) - (4.49) and (4.33) - (4.36), we consider the 2-dimensional circular limit cycle system of Section 5.7.3. The rates of convergence of the two schemes is illustrated in Figure 4.2.



(a) Relative Error

(b) Absolute Error

Figure 4.2: Comparison between Schemes using Interpolation Times $\delta_T(x)$ and δ_T

4.5 Accelerating Convergence of the Available Power

In practical applications of the centered schemes (4.33)-(4.36), (4.46)-(4.49), it is often the case that the rate of convergence of the available power approximation $\lambda^\delta_{a,k}$ is quite slow compared with the rate of convergence of the value function. This is demonstrated in Figure 4.3, which compares the two rates for the 2-dimensional circular limit cycle system of Section 5.7.3.

In order to understand the dynamics of the available power approximation $\lambda^\delta_{a,k}$, apply (4.48) so that (4.49) becomes

$$\begin{aligned}
\lambda^\delta_{a,k+1} &= \frac{R^\delta_k + \tilde{V}^\delta_{k+1}(x_0)}{(k+1)\delta_T(x_0)} \\
&= \left(\frac{k}{k+1}\right)\frac{R^\delta_k}{k\delta_T(x_0)} + \frac{1}{k+1}\left(\frac{\tilde{V}^\delta_{k+1}(x_0)}{\delta_T(x_0)}\right) \\
&= \left(\frac{k}{k+1}\right)\lambda^\delta_{a,k} + \frac{1}{k+1}\left(\frac{\tilde{V}^\delta_{k+1}(x_0)}{\delta_T(x_0)}\right).
\end{aligned}$$

Hence, the available power approximation dynamics arise from the discrete time averaging filter

$$z_{k+1} = \left(\frac{k}{k+1}\right) z_k + \left(\frac{1}{k+1}\right) v_{k+1}, \quad z_0 = 0, \tag{4.51}$$

where $\lambda^\delta_{a,k} = z_k$, and $v_k = \frac{\tilde{V}^\delta_k(x_0)}{\delta_T(x_0)}$. So, the approximation of the available power

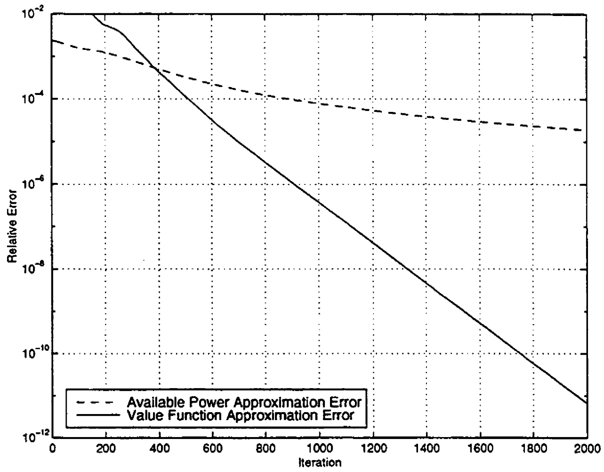


Figure 4.3: Typical Convergence of $\lambda^{\delta}_{a,k}$

is performed in open-loop, as illustrated in Figure 4.4. That is, the available power approximation relies on the approximation of the value function, but not vice-versa. Furthermore, since (4.51) describes an averaging process, (4.51) has a damping effect on the input v_k. Hence, the convergence of the available power approximation (4.49) can be hampered by the averaging filter (4.51).

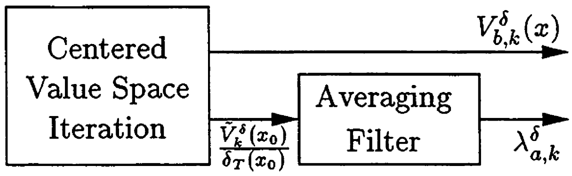


Figure 4.4: Block Diagram for the Centered Schemes

If it is assumed that $\tilde{V}^{\delta}_k$ converges to some $\tilde{V}^{\delta}_{\infty}$ as $k \to \infty$, inspection of (4.51) reveals that the "steady state" available power approximation is given by

$$\lambda^{\delta}_{a,\infty} = \frac{\tilde{V}^{\delta}_{\infty}(x_0)}{\delta_T(x_0)}. \tag{4.52}$$

Hence, from the definition of $\tilde{V}^{\delta}_k$ (4.27), notation (4.39), and the finite difference approximation (4.20) for the partial derivative $\frac{\partial V}{\partial T}(x,T)$,

$$\begin{aligned}\lambda^{\delta}_{a,\infty} &= \frac{\tilde{V}^{\delta}_{\infty}(x_0)}{\delta_T(x_0)} \\ &= \lim_{k\to\infty}\left\{\frac{V^{\delta}_k(x_0)-V^{\delta}_{k-1}(x_0)}{\delta_T(x_0)}\right\}\end{aligned}$$

$$\approx \lim_{T\to\infty} \left\{ \frac{\partial V}{\partial T}(x,T) \right\}. \tag{4.53}$$

But, this is as expected from Theorem 2.6.7. That is, rather than approximating the available power by approximating the ratio $\frac{V(x,T)}{T}$, where $V(x,T)$ is the finite horizon value function (2.33), the partial derivative $\frac{\partial V}{\partial T}(x,T)$ may be approximated. This results in the following revised centered finite difference scheme:

$$\begin{aligned}
\tilde{V}^{\delta}_{k+1}(x) &= \max_{v\in G_V} \left\{ \sum_{z\in N^{\delta}X(x)} \left[\tilde{V}^{\delta}_{b,k}(z)p(x,z|v)\right] + \delta_T(x)\left[c(x)-\gamma^2|v|^2\right] \right\}, && (4.54)\\
\tilde{V}^{\delta}_{b,k+1}(x) &= \tilde{V}^{\delta}_{k+1}(x) - \tilde{V}^{\delta}_{k+1}(x_0)\left(\frac{\delta_T(x)}{\delta_T(x_0)}\right), && (4.55)\\
\lambda^{\delta}_{a,k} &= \frac{\tilde{V}^{\delta}_{k}(x_0)}{\delta_T(x_0)}, && (4.56)
\end{aligned}$$

with initialization (4.37). A comparison of schemes (4.33)-(4.36) and (4.54)-(4.56) (for the minimal interpolation time δ_T (4.16)) is provided by way of the 2-dimensional circular limit cycle system of Section 5.7.3. Note that the value space iteration is identical for the two schemes. The results are illustrated in Figure 4.5. As expected, the available power approximation $\lambda^{\delta}_{a,k}$ (4.56) (the "Accelerated Available Power Approximation" in Figure 4.5) converges at the same rate as the value function (4.55).

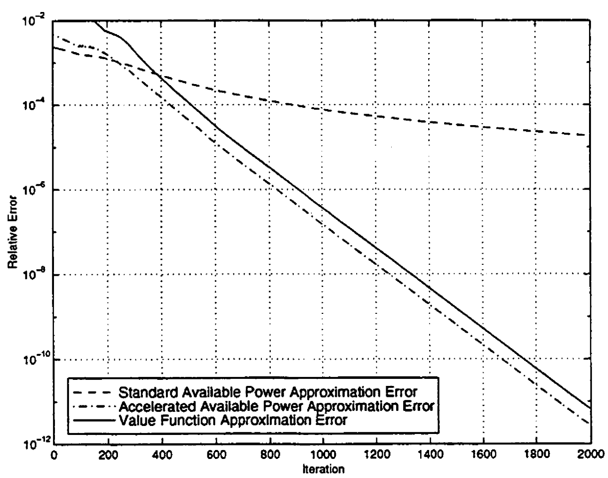


Figure 4.5: Accelerated Convergence of $\lambda^{\delta}_{a,k}$

Note that it is possible to obtain a continuum of approximations $\lambda^{\delta}_{a,k}$ between the two approximations (4.49) and (4.56). Writing the averaging filter (4.51) as

$$(k+1)z_{k+1} = kz_k + v_{k+1},$$

it is possible to increase the speed of convergence of z_k without affecting the steady state value by moving the pole of the filter closer to the unit circle. This is achieved via the modified filter

$$(k+L)z_{k+1} = kz_k + Lv_{k+1}, \tag{4.57}$$

where $L \geq 1$. With $v_k = \frac{\tilde{V}_k^\delta}{\delta_T(x_0)}$ and $\lambda_{a,k}^\delta = z_k$, $L = 1$ recovers the approximation (4.49), whilst $L = \infty$ recovers the approximation (4.56). For intermediate values of L, the corresponding centered finite difference scheme is given by

$$\hat{V}_{k+1}^\delta(x) = \max_{v \in G_V} \left\{ \sum_{z \in N^\delta X(x)} \left[\hat{V}_{b,k}^\delta(z) p(x,z|v) \right] + \delta_T(x) \left[c(x) - \gamma^2 |v|^2 - \left(1 - \frac{1}{L}\right) \lambda_{a,k}^\delta \right] \right\}, \tag{4.58}$$

$$\hat{V}_{b,k+1}^\delta(x) = \hat{V}_{k+1}^\delta(x) - \hat{V}_{k+1}^\delta(x_0) \left(\frac{\delta_T(x)}{\delta_T(x_0)} \right), \tag{4.59}$$

$$R_{k+1}^\delta = R_k^\delta + \hat{V}_{k+1}^\delta(x_0), \tag{4.60}$$

$$\lambda_{a,k}^\delta = \left(\frac{Lk}{L-1+k} \right) \left(\frac{R_k^\delta}{k\delta_T(x_0)} \right), \tag{4.61}$$

with initialization (4.37) and (4.38). Note that this scheme employs a feedback of the current available power approximation $\lambda_{a,k}^\delta$ (4.61) into the value space iteration (4.58). However, by simple manipulations of equations (4.58)-(4.61), it will be shown that this scheme is still of the form of Figure 4.4, with the averaging filter given by (4.57).

Recall that the value space iteration for the scheme (4.46)-(4.49) is given by

$$\tilde{V}_{k+1}^\delta(x) = \max_{v \in G_V} \left\{ \sum_{z \in N^\delta X(x)} \left[\tilde{V}_{b,k}^\delta(z) p(x,z|v) \right] + \delta_T(x) \left[c(x) - \gamma^2 |v|^2 \right] \right\}. \tag{4.62}$$

Then, rewriting (4.58),

$$\hat{V}_{k+1}^\delta(x) = \tilde{V}_{k+1}^\delta(x) - \delta_T(x) \left(1 - \frac{1}{L}\right) \lambda_{a,k}^\delta. \tag{4.63}$$

Hence, the approximation for $V_b(x) - V_b(x_0)$ (4.59) becomes

$$\begin{aligned}
\hat{V}_{b,k+1}^\delta(x) &= \hat{V}_{k+1}^\delta(x) - \hat{V}_{k+1}^\delta(x_0) \left(\frac{\delta_T(x)}{\delta_T(x_0)} \right) \\
&= \tilde{V}_{k+1}^\delta(x) - \delta_T(x) \left(1 - \frac{1}{L}\right) \lambda_{a,k}^\delta - \tilde{V}_{k+1}^\delta(x_0) \left(\frac{\delta_T(x)}{\delta_T(x_0)} \right) + \\
&\qquad \delta_T(x_0) \left(1 - \frac{1}{L}\right) \left(\frac{\delta_T(x)}{\delta_T(x_0)} \right) \lambda_{a,k}^\delta \\
&= \tilde{V}_{k+1}^\delta(x) - \tilde{V}_{k+1}^\delta(x_0) \left(\frac{\delta_T(x)}{\delta_T(x_0)} \right),
\end{aligned} \tag{4.64}$$

which is precisely the centering step (4.47). Furthermore, since $\tilde{V}_{k+1}^\delta(x)$ depends on $\tilde{V}_{b,k}^\delta$ only, the approximations for $V_b(x) - V_b(x_0)$ must be identical for schemes (4.46)-(4.49) and (4.58)-(4.61) for each k. That is, $\hat{V}^{\delta}_{b,k}(x) \equiv \tilde{V}^{\delta}_{b,k}(x)$ for all $x \in G_X$ and all iterations k.

Mutliplying (4.60) through by L, substituting for LR^{δ}_{k+1} and LR^{δ}_{k} using (4.61), and applying (4.63),

$$\begin{aligned}
(L+k)\lambda^{\delta}_{a,k+1}\delta_T(x_0) &= (L-1+k)\lambda^{\delta}_{a,k}\delta_T(x_0) + L\hat{V}^{\delta}_{k+1}(x_0) \\
&= (L-1+k)\lambda^{\delta}_{a,k}\delta_T(x_0) + L\left(\tilde{V}^{\delta}_{k+1}(x_0) - \delta_T(x)\left(1-\frac{1}{L}\right)\lambda^{\delta}_{a,k}\right) \\
&= (L-1+k)\lambda^{\delta}_{a,k}\delta_T(x_0) + L\tilde{V}^{\delta}_{k+1}(x_0) - (L-1)\lambda^{\delta}_{a,k} \\
&= k\lambda^{\delta}_{a,k}\delta_T(x_0) + L\tilde{V}^{\delta}_{k+1}(x_0).
\end{aligned}$$

Hence,

$$(L+k)\lambda^{\delta}_{a,k+1} = k\lambda^{\delta}_{a,k} + L\left(\frac{\tilde{V}^{\delta}_{k+1}(x_0)}{\delta_T(x_0)}\right), \tag{4.65}$$

which is precisely (4.57). Hence, the modified finite difference scheme (4.58)-(4.61) is identical to the scheme (4.46)-(4.49) with the averaging filter (4.51) replaced by the filter (4.57). Note that setting $L = 1$ in the scheme (4.58)-(4.61) yields the scheme (4.46)-(4.49). Setting $L = \infty$ in (4.58) yields the iteration

$$\hat{V}^{\delta}_{k+1}(x) = \tilde{V}^{\delta}_{k+1}(x) - \delta_T(x)\lambda^{\delta}_{a,k}, \tag{4.66}$$

where $\tilde{V}^{\delta}_{k+1}(x)$ is given by (4.62). Therefore, (4.60) implies that

$$R^{\delta}_{k+1} = R^{\delta}_{k} + \tilde{V}^{\delta}_{k+1}(x_0) - \delta_T(x_0)\lambda^{\delta}_{a,k}. \tag{4.67}$$

But, $L = \infty$ in (4.61) implies that

$$\lambda^{\delta}_{a,k} = \frac{R^{\delta}_{k}}{\delta_T(x_0)}. \tag{4.68}$$

Substituting this in (4.67),

$$\begin{aligned}
R^{\delta}_{k+1} &= R^{\delta}_{k} + \tilde{V}^{\delta}_{k+1}(x_0) - R^{\delta}_{k} \\
&= \tilde{V}^{\delta}_{k+1}(x_0).
\end{aligned}$$

Hence, (4.68) implies that

$$\lambda^{\delta}_{a,k} = \frac{\tilde{V}^{\delta}_{k}(x_0)}{\delta_T(x_0)}, \tag{4.69}$$

which is precisely (4.56). Hence, setting $L = \infty$ in the scheme (4.58)-(4.61) yields the scheme (4.54)-(4.56) (noting that the centering step (4.59) is independent of L, as shown in (4.64), so that the k^{th} approximation for $V_b(x) - V_b(x_0)$ is identical for both schemes).

The effect of varying $L \geq 1$ is illustrated in Figure 5.43 (for the 2-dimensional circular limit cycle system of Section 5.7.3).

4.6 A Bisection Method for (λ_a, V_b)

One of the main problems with the centered finite difference schemes presented thus far is that the value function approximation generated is for the normalized infinite horizon available storage, $V_b(x) - V_b(x_0)$, rather than the unnormalized infinite horizon available storage $V_b(x)$, (2.169). Although the reference state $x_0 \in G_X$ is arbitrary, clearly the normalized approximation must satisfy $\tilde{V}^{\delta}_{b,k}(x_0) = 0$. Since the actual value of the infinite horizon value function $V_b(x)$ is unknown at x_0, $V_b(x)$ cannot be recovered from the centered finite difference approximation. Hence, the centered schemes presented are useful for computing the shape, but not the value, of the function $V_b(x)$. Clearly, such an approximation may be unsatisfactory. For example, it does not admit the testing of results such as Corollary 2.10.3 and Corollary 2.10.4.

In this section, an alternative to the centered schemes is presented. The bisection finite difference scheme allows the approximation of the infinite horizon value function $V_b(x)$ to be computed, rather than the normalization $V_b(x) - V_b(x_0)$. Specifically, the bisection approach approximates the function $V(x,T) - \lambda_a T$ rather than the normalization $V(x,T) - V(x_0,T)$ (as is the case with the centered schemes), so that as $T \to \infty$, the approximation tends to $V_b(x)$ rather than $V_b(x) - V_b(x_0)$. Note that the term "bisection" refers to the method by which the available power approximation is computed.

A natural starting point for developing the bisection scheme is to consider the approximation for $V(x,T) - \lambda_a T$ given by

$$V_b^{\delta}(x,T) \;=\; V^{\delta}(x,T) - \lambda_a^{\delta} T, \tag{4.70}$$

where $V^{\delta}(x,T)$ is the finite difference approximation of the finite horizon value function $V(x,T)$ (4.5) obtained using (4.40), and λ_a^{δ} is an as yet to be determined approximation of the available power λ_a (4.1). Combining (4.40) and (4.70),

$$\begin{aligned} & V_b^{\delta}(x, T + \delta_T(x)) \\ &= \; V^{\delta}(x, T + \delta_T(x)) - \lambda^{\delta}(T + \delta_T(x)) \end{aligned}$$

$$= \max_{v \in G_V} \left\{ \sum_{z \in N^{\delta_X}(x)} \left[\left(V^\delta(z,T) - \lambda^\delta T \right) p(x,z|v) \right] + \delta_T(x) \left[c(x) - \gamma^2 |v|^2 - \lambda_a^\delta \right] \right\}$$

$$= \max_{v \in G_V} \left\{ \sum_{z \in N^{\delta_X}(x)} \left[V_b^\delta(z,T) p(x,z|v) \right] + \delta_T(x) \left[c(x) - \gamma^2 |v|^2 - \lambda_a^\delta \right] \right\}. \qquad (4.71)$$

Applying assumption (A19),

$$V_b^\delta(x, k\delta_T(x)) \approx V_b^\delta(z, k\delta_T(x)) \qquad (4.72)$$

for $z \in N^\delta(x)$ and large k. Hence, writing

$$V_{b,k}^\delta(x) = V_b^\delta(x, k\delta_T(x)), \qquad (4.73)$$

equations (4.71), (4.72), and (4.73) imply that

$$V_{b,k+1}^\delta(x) = \max_{v \in G_V} \left\{ \sum_{z \in N^{\delta_X}(x)} \left[V_{b,k}^\delta(z) p(x,z|v) \right] + \delta_T(x) \left[c(x) - \gamma^2 |v|^2 - \lambda_a^\delta \right] \right\}. \qquad (4.74)$$

Since λ_a^δ is an approximation for the available power λ_a (4.1), the approximation $V_{b,k}^\delta(x)$ produced by the iterative scheme (4.74) is an approximation for the infinite horizon λ-storage $V_{b\lambda_a^\delta}(x)$ (2.192) rather than the infinite horizon available storage $V_b(x)$ (4.3). Theorem 2.11.2 states that

$$V_{b\lambda_a^\delta}(x) = \begin{cases} -\infty & \lambda_a^\delta > \lambda_a, \\ V_b(x) & \lambda_a^\delta = \lambda_a, \\ \infty & \lambda_a^\delta < \lambda_a. \end{cases}$$

Hence, it is reasonable to expect that

(i) $V_{b,k}^\delta(x)$ diverges to $-\infty$ if $\lambda_a^\delta > \lambda_a$,

(ii) $V_{b,k}^\delta(x)$ converges to $V_b(x)$ if $\lambda_a^\delta = \lambda_a$,

(iii) $V_{b,k}^\delta(x)$ diverges to $+\infty$ if $\lambda_a^\delta < \lambda_a$,

in the limit as $k \to \infty$ and δ_X, $\delta_V \downarrow 0$. This suggests the following bisection approach for approximating the pair $(\lambda_a, V_b(x))$:

Step 1. Set $j = 0$. Guess $\lambda_{a,j}^{\delta,l}$, $\lambda_{a,j}^{\delta,u}$, where $\lambda_{a,j}^{\delta,l} \le \lambda_a \le \lambda_{a,j}^{\delta,u}$.

Step 2. Set $\lambda_a^{\delta,m} = \frac{\lambda_{a,j}^{\delta,l}+\lambda_{a,j}^{\delta,u}}{2}$. Set $k = 0$. Set $V_{b,k}^{\delta}(x) = 0$ for all $x \in G_X$.

Step 3. Compute $V_{b,k+1}^{\delta}(x)$ using the value space iteration (4.74). Repeat Step 3 until $V_{b,k}^{\delta}(x)$ diverges or converges to within a desired tolerance.

Step 4. If the result of Step 3 is divergent to $-\infty$ then set $\lambda_{a,j+1}^{\delta,u} = \lambda_a^{\delta,m}$, else set $\lambda_{a,j+1}^{\delta,l} = \lambda_a^{\delta,m}$.

Step 5. Repeat Steps 2-4 until $\lambda_{a,j}^{\delta,u} - \lambda_{a,j}^{\delta,l}$ is within a desired tolerance.

In summary, this scheme computes the infinite horizon λ-storage $V_{b,\lambda}(x)$ using the value space iteration (4.74) with λ_a^δ fixed. If the result is divergent to $-\infty$, then the choice of λ_a^δ is too large. Otherwise, if the result is divergent to ∞, then the choice of λ_a^δ is too small.

Clearly this scheme is numerically very intensive. Furthermore, the test in Step 3 for divergence or convergence can never be absolutely conclusive since only a finite number of value iterations can be performed. As such, the usefulness of the bisection scheme for approximating both the available power λ_a and the infinite horizon available storage V_b is limited. However, in computing $V_b(x)$ with λ_a known, the value space iteration (4.74) is very useful. Even without knowledge of the exact available power, a centered scheme such as that presented in Section 4.3 can provide an accurate approximation of the available power, which may then be used in (4.74). Although using a centered scheme first may appear to defeat the purpose, it is important to remember that centered schemes can only compute the normalization $V_b(x) - V_b(x_0)$ (for some reference state x_0). The purpose of the bisection scheme is to compute the unnormalized infinite horizon available storage $V_b(x)$.

Remark 4.6.1 Value space iteration (4.74) is clearly of a similar form to (4.58), since both include the available power approximation in the running cost term.

Preserving the $\lambda_{a,k}^\delta$ feedback in the scheme (4.58)-(4.61), consider again the case where $L = \infty$. The available power approximation (4.61) is given then by (4.69),

$$\lambda_{a,k}^{\delta} = \frac{\tilde{V}_k^{\delta}(x_0)}{\delta_T(x_0)}.$$

Furthermore, the value space iteration (4.58) is given by (4.66),

$$\hat{V}_{k+1}^{\delta}(x) = \tilde{V}_{k+1}^{\delta}(x) - \delta_T(x)\lambda_{a,k}^{\delta}.$$

Combining these two equations,

$$\hat{V}_{k+1}^{\delta}(x) = \tilde{V}_{k+1}^{\delta}(x) - \tilde{V}_k^{\delta}(x_0)\left(\frac{\delta_T(x)}{\delta_T(x_0)}\right).$$

For a convergent approximation, clearly $\tilde{V}_k^{\delta}(x) \to \tilde{V}_{k+1}^{\delta}(x)$ for all $x \in G_X$. So, for large k, the above value space iteration can be approximated by

$$\begin{aligned}\hat{V}_{k+1}^{\delta}(x) &\approx \tilde{V}_{k+1}^{\delta}(x) - \tilde{V}_{k+1}^{\delta}(x_0)\left(\frac{\delta_T(x)}{\delta_T(x_0)}\right) \\ &= V_{b,k+1}^{\delta}(x), \qquad (4.75)\end{aligned}$$

from the centering step (4.59) (which is independent of L as shown in (4.64)). So, combining approximation (4.75) with the value space iteration (4.58) yields an alternative value space iteration,

$$V_{b,k+1}^{\delta}(x) = \max_{v \in G_V}\left\{ \sum_{z \in N^{\delta X}(x)}\left[V_{b,k}^{\delta}(z)p(x,z|v)\right] + \delta_T(x)\left[c(x) - \gamma^2|v|^2 - \lambda_{a,k}^{\delta}\right]\right\}, \qquad (4.76)$$

which is the same form as (4.74). ◀

Remark 4.6.2 The value space iteration (4.74) can also be derived by proposing that (λ_a, V_b) is a solution of the stationary form of the PDE (2.50). That is, that

$$\lambda_a = \sup_{v \in \mathbf{R}^p}\left\{\nabla_x V_b(x)\cdot g(x,v) + c(x) - \gamma^2|v|^2\right\}, \qquad (4.77)$$

where $g(x,v) = a(x) + b(x)v$ is the RHS of the state equation for system Σ (2.1). Following the same steps as in Section 4.3, the finite difference approximation for $\nabla_x V_b(x)\cdot g(x,v)$ is given by

$$\nabla_x^{\delta X} V_b(x)\cdot g(x,v) = \frac{1}{\delta_T(x)}\left\{-V_b(x) + \sum_{z \in N^{\delta X}(x)}\left[V_b(z)p(x,z|v)\right]\right\}, \qquad (4.78)$$

where $p(x,z|v)$ is given by (4.50). Limiting the disturbance space to the coordinate grid G_V and applying the approximation (4.78) to the stationary PDE (4.77),

$$\lambda_a^{\delta} = \max_{v \in G_V}\left\{\frac{1}{\delta_T(x)}\left\{-V_b^{\delta}(x) + \sum_{z \in N^{\delta X}(x)}\left[V_b^{\delta}(z)p(x,z|v)\right]\right\} + c(x) - \gamma^2|v|^2\right\}.$$

where the δ superscripts denote that the quantities are approximate. Rearranging yields that

$$V_b^\delta(x) = \max_{v \in G_V} \left\{ \sum_{z \in N^\delta X(x)} \left[V_b^\delta(z) p(x, z|v) \right] + \delta_T(x) \left[c(x) - \gamma^2 |v|^2 - \lambda_a^\delta \right] \right\}, \quad (4.79)$$

which defines a recursive expression for the infinite horizon available storage approximation $V_b^\delta(x)$. Setting $V_{b,k+1}^\delta(x)$ equal to $V_b^\delta(x)$ and $V_{b,k}^\delta(z)$ equal to $V_b^\delta(z)$ yields the value space iteration (4.74). ◀

4.7 A Bisection Method for (λ_a, V_a)

Provided that assumption (A19) holds, the infinite horizon available storage $V_b(x)$ (4.3) can be defined as the limit as $T \to \infty$ of the value function $V(x,T) - \lambda_a T$. That is, $V_b(x)$ may be regarded as the stationary value of $V(x,T) - \lambda_a T$. This allows approximation schemes such as those of Section 4.3 to focus on the approximation of the finite horizon value function $V(x,T)$, where $T \to \infty$ is achieved approximately as the iteration number $k \to \infty$. That is, the approximation for $V_b(x)$ using finite differences may be constructed from the nonstationary PDE (4.7) for $V(x,T)$.

The main difficulty with constructing similar schemes for approximating the super available storage $V_a(x)$ (4.2) is that the representation of $V_a(x)$ in terms of the finite horizon value function $V(x,T)$ involves a supremum over the time horizon T. That is, $V_a(x)$ cannot be interpreted as a stationary solution of the nonstationary PDE (4.7). However, by Theorem 2.8.12, is it known that the pair (λ_a, V_a) must satisfy the stationary variational inequality (2.163). That is,

$$\max \left(-\lambda_a + \sup_{v \in \mathbf{R}^p} \left\{ \nabla_x V_a(x) \cdot g(x,v) + c(x) - \gamma^2 |v|^2 \right\}, -V_a(x) \right) = 0, \quad (4.80)$$

where $g(x,v) = a(x) + b(x)v$ is the RHS of the state equation (2.1). Furthermore, in developing a bisection method for the pair (λ_a, V_b), Remark 4.6.2 demonstrates that the value space iteration for the infinite horizon available storage approximation can be derived from the stationary PDE (4.77). Hence, to obtain a value space iteration for an approximation to $V_a(x)$, the same method as in Remark 4.6.2 is applied using the stationary variational inequality (4.80).

Following steps similar to those in Section 4.3, the finite difference approximation for $\nabla_x V_a(x) \cdot g(x,v)$ can be written as

$$\nabla_x^{\delta X} V_a(x) \cdot g(x,v) = \frac{1}{\delta_T(x)} \left\{ -V_a(x) + \sum_{z \in N^\delta(x)} [V_a(z) p(x,z|v)] \right\}, \quad (4.81)$$

where $p(x,z|v)$ is given by (4.50). Once again limiting the disturbance space to the coordinate grid G_V and applying the approximation (4.81) to the stationary VI (4.80),

$$\begin{aligned} \max\Bigg(& \\ & -\lambda_a^\delta + \max_{v \in G_V} \left\{ \frac{1}{\delta_T(x)} \left\{ -V_a^\delta(x) + \sum_{z \in N^{\delta X}(x)} \left[V_a^\delta(z) p(x,z|v) \right] \right\} + c(x) - \gamma^2 |v|^2 \right\}, \\ & -V_a^\delta(x) \\ \Bigg) &= 0, \end{aligned} \quad (4.82)$$

where the δ superscript implies an approximation. Multiplying through by $\delta_T(x)$, adding $V_a(x)$ to both sides, and introducing the iteration number k yields the required value space iteration,

$$\begin{aligned} V_{a,k+1}^\delta(x) = \max\Bigg(& \\ & \max_{v \in G_V} \left\{ \sum_{z \in N^{\delta X}(x)} \left[V_{a,k}^\delta(z) p(x,z|v) \right] + \delta_T(x) \left[c(x) - \gamma^2 |v|^2 - \lambda_a^\delta \right] \right\}, \\ & V_{a,k}^\delta(x) \left[1 - \delta_T(x) \right] \\ \Bigg), & \end{aligned} \quad (4.83)$$

with initial value $V_{a,k=0}^\delta(x) = 0$ for all $x \in G_X$. Note that since λ_a^δ is an as yet to be determined approximation for the available power λ_a, the approximation $V_{a,k}^\delta(x)$ given by the value space iteration (4.83) is actually an approximation to the super λ-storage $V_{a\lambda_a^\delta}(x)$ (2.164). However, Theorem 2.9.2 states that

$$\begin{aligned} \lambda_a^\delta > \lambda_a &\Rightarrow V_{a\lambda_a^\delta}(x) \le V_a(x), \\ \lambda_a^\delta = \lambda_a &\Rightarrow V_{a\lambda_a^\delta}(x) = V_a(x), \\ \lambda_a^\delta < \lambda_a &\Rightarrow V_{a\lambda_a^\delta}(x) = \infty. \end{aligned}$$

Hence, similar to the infinite horizon available storage case of Section 4.6, it is reasonable to expect that

(i) $V_{a,k}^\delta(x)$ converges to a lower bound for $V_a(x)$ if $\lambda_a^\delta > \lambda_a$,

(ii) $V_{a,k}^\delta(x)$ converges to $V_a(x)$ if $\lambda_a^\delta = \lambda_a$,

(iii) $V^{\delta}_{a,k}(x)$ diverges to ∞ if $\lambda^{\delta}_{a} < \lambda_a$,

in the limit as $k \to \infty$ and δ_X, $\delta_V \downarrow 0$. This suggests the following bisection scheme for approximating the pair (λ_a, V_a), very similar to that of Section 4.6:

Step 1. Set $j = 0$. Guess $\lambda^{\delta,l}_{a,j}$, $\lambda^{\delta,u}_{a,j}$, where $\lambda^{\delta,l}_{a,j} \leq \lambda_a \leq \lambda^{\delta,u}_{a,j}$.

Step 2. Set $\lambda^{\delta,m}_{a} = \frac{\lambda^{\delta,l}_{a,j} + \lambda^{\delta,u}_{a,j}}{2}$. Set $k = 0$. Set $V^{\delta}_{a,k}(x) = 0$ for all $x \in G_X$.

Step 3. Compute $V^{\delta}_{a,k+1}(x)$ using the value space iteration (4.83). Repeat Step 3 until $V^{\delta}_{a,k}(x)$ diverges or converges to within a desired tolerance.

Step 4. If the result of Step 3 is divergent to ∞ then set $\lambda^{\delta,l}_{a,j+1} = \lambda^{\delta,m}_{a}$, else set $\lambda^{\delta,u}_{a,j+1} = \lambda^{\delta,m}_{a}$.

Step 5. Repeat Steps 2-4 until $\lambda^{\delta,u}_{a,j} - \lambda^{\delta,l}_{a,j}$ is within a desired tolerance.

4.8 A Centered Method for $\mathbf{V}^{\mathbf{f}}_{\mathbf{br}}(\xi, \mathbf{x})$

Unlike standard $\mathcal{H}_\infty$ theory, the existence of a minimum in either the super available storage $V_a(x)$ (4.2) or the infinite horizon available storage $V_b(x)$ (4.3) for systems with power gain does not necessarily imply the existence of a stable equilibrium at that minimum (see, for example, Figure 5.6, where the stable equilibrium under the worst case disturbance is actually $\frac{1}{3}$). However, as discussed in Section 2.14, it is evident that the existence of a minimum in the difference between the infinite horizon fixed initial state required supply and the infinite horizon available storage does imply a form of stability of the worst case trajectory. Consequently, in order to calculate this difference $W(x) = V^{f}_{br}(\xi, x) - V_b(x)$ (2.211) for a given initial state $\xi = \text{argmin}_{x \in \mathbf{R}^n}\{V_b(x)\}$, an approximation for $V^{f}_{br}(\xi, x)$ is required.

In Section 4.3, an approximation for the finite horizon value function $V(x, T)$ (4.5) was computed by applying a finite difference approximation to the nonstationary PDE (2.50). This approximation of $V(x, T)$ was then applied in the definition (4.3) of the infinite horizon available storage (4.3) to compute an approximation for $V_b(x)$. Following the same procedure for $V^{f}_{br}(\xi, x)$, a finite difference approximation can be applied to the nonstationary PDE (2.198), yielding an approximation for the finite horizon fixed initial state required supply $V_r(\xi, x, T)$ (2.194). Using this approximation, it is then possible to construct a centered finite difference approximation for the infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$ by applying definition (4.4). As the details of this procedure are largely similar to that of Section 4.3, only the resulting centered finite difference method for $V_{br}^f(\xi, x)$ is presented. Note that as an aside from the approximation of $V_{br}^f(\xi, x)$, an approximation for the available power λ_a can also be computed. However, this is not essential as λ_a may be computed when approximation $V_b(x)$ (Section 4.3).

As a matter of notation, let G_X and G_V denote the restricted state space and disturbance coordinate grids (4.11), (4.12) respectively. Also, let $\tilde{V}^\delta_{br,\xi,k}(x)$ denote the k^{th} approximation of the normalized infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$ (4.4). That is,

$$\tilde{V}^\delta_{br,\xi,k}(x) \approx V_{br}^f(\xi, x) - V_{br}^f(\xi, x_0), \tag{4.84}$$

where $x_0 \in G_X$ is an arbitrary fixed reference state. Then, a centered finite difference method for approximating $V_{br}^f(\xi, x)$ is given by

$$\begin{aligned} \tilde{V}^\delta_{r,\xi,k+1}(x) &= \min_{v \in G_V} \left\{ \sum_{z \in N^{\delta_X}(x)} \left[\tilde{V}_{br,\xi,k}(z) \bar{p}(x, z|v) \right] + \gamma^2 |v|^2 - c(x) \right\}, && (4.85) \\ \tilde{V}^\delta_{br,\xi,k+1}(x) &= \tilde{V}^\delta_{r,\xi,k+1}(x) - \tilde{V}^\delta_{r,\xi,k+1}(x_0), && (4.86) \end{aligned}$$

where

$$\bar{p}(x, z|v) = \begin{cases} 1 - \frac{|g(x,v)|_1}{m} & z = x, \\ \frac{g_i^{\pm}(x,v)}{m} & z = x \mp \delta_X e_i, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.87}$$

In deriving value space iteration (4.85) and centering equation (4.86), the only notable difference from the corresponding derivation in Section 4.3 is the choice of finite difference approximation. In this case, the i^{th} component of the finite difference approximation is (c.f. (4.8))

$$\nabla^{\delta_X}_{x,i} V_r(\xi, x, T) = \begin{cases} \frac{V_r(\xi, x+\delta_X e_i, T) - V_r(\xi, x, T)}{\delta_X} & g_i(x, v) < 0, \\ \frac{V_r(\xi, x, T) - V_r(\xi, x-\delta_X e_i, T)}{\delta_X} & g_i(x, v) \geq 0. \end{cases} \tag{4.88}$$

The centered finite difference method (4.85)-(4.86) is applied in several examples in Chapter 5 (see, for example, Section 5.3.3, 5.4).

4.9 Conclusion

By applying finite difference approximations to the various deterministic differential equations for the candidate power bias / storage function pairs, it was possible to derive useful approximation methods for these candidate pairs.

For the infinite horizon available storage $V_b(x)$ and infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$, treatment of the nonstationary partial differential equations allowed approximation of the corresponding finite horizon value functions. These approximations when used in the limit of large time horizons then provided approximations for the infinite horizon functions $V_b(x)$ and $V_{br}^f(\xi, x)$.

By applying a finite difference approximation to the stationary variational inequality for the super available storage $V_a(x)$, a similar approximation method for the super available storage $V_a(x)$ was developed.

Finally, two methods for approximating the available power λ_a were presented. By far the most useful method employed a "centering" technique to compute the growth in the finite horizon value function $V(x, T)$ with T, and hence the worst case average cost per unit time λ_a.

Chapter 5

Examples

5.1 Static Nonlinearities

In this section, we consider the problem of calculating the available storage for static nonlinearities (or just plain functions). Hence, we consider systems of the form

$$\Sigma \ : \ z = \Delta(v), \tag{5.1}$$

where $v \in \mathbf{R}^p$, $z \in \mathbf{R}^q$, and $\Delta(\cdot)$ is a function mapping R^p to R^q. As such, a static nonlinearity does not require an internal state, although it could be written as

$$\Sigma \ : \ \begin{cases} \dot{x}(t) &= 0, \ x(0) = x, \\ z(t) &= \Delta(v(t)). \end{cases}$$

As such, there is no associated energy storage with static nonlinearities. So, calculating the available storage for static nonlinearities is a relatively simple task, especially in view of Theorem 2.10.15. Since the energy storage can be regarded as constant, finding a solution pair (λ, V) of the PDE (2.179) simplifies to computing

$$\lambda \ = \ H(x, 0). \tag{5.2}$$

Theorem 2.10.15 ensures that the power bias computed is the available power. That is, $\lambda_a = \lambda$. Hence, for static nonlinearities $\Delta(\cdot)$,

$$\lambda_a \ = \ \sup_{v \in \mathbf{R}^p} \left\{ |\Delta(v)|^2 - \gamma^2 |v|^2 \right\}. \tag{5.3}$$

Note that (5.3) also follows directly from the definition of available power (2.84), where the supremum over $v \in \mathcal{L}_2[0,T]$ simplifies to a supremum over $v \in \mathbf{R}^p$ due to the absence of dynamics in the system.

Given the prescription (5.3), we compute the available power for a number examples static nonlinearities.

Example 5.1.1 Define a unity gain deadzone nonlinearity with deadband $[-1, 1]$ as

$$\delta(v) = \begin{cases} v+1 & v < -1, \\ 0 & v \in [-1, 1], \\ v-1 & v > 1. \end{cases} \tag{5.4}$$

Since the deadzone is piecewise continuous, it is useful to compute the supremum in (5.3) on each on the three intervals on which $\delta(\cdot)$ (5.4) is defined. That is,

$$\lambda^*_{(-\infty,-1)} = \sup_{v \in (-\infty,-1)} \left\{ |\delta(v)|^2 - \gamma^2 |v|^2 \right\}, \tag{5.5}$$

$$\lambda^*_{[-1,1]} = \sup_{v \in [-1,1]} \left\{ |\delta(v)|^2 - \gamma^2 |v|^2 \right\}, \tag{5.6}$$

$$\lambda^*_{(1,\infty)} = \sup_{v \in (1,\infty)} \left\{ |\delta(v)|^2 - \gamma^2 |v|^2 \right\}. \tag{5.7}$$

Then, the available power (5.3) is given by

$$\lambda_a = \max \left\{ \lambda^*_{(-\infty,-1)}, \lambda^*_{[-1,1]}, \lambda^*_{(1,\infty)} \right\}. \tag{5.8}$$

So, applying (5.4) to (5.5),

$$\begin{aligned} \lambda^*_{(-\infty,-1)} &= \sup_{v \in (-\infty,-1)} \left\{ |v+1|^2 - \gamma^2 |v|^2 \right\} \\ &= \sup_{v \in (-\infty,-1)} \left\{ 1 + 2v + (1-\gamma^2) v^2 \right\} \end{aligned}$$

Since the supremum is over an unbounded set, for finiteness of $\lambda^*_{(-\infty,-1)}$, we require that $\gamma > 1$. In this case, the maximum occurs when

$$v = \bar{v} = \frac{1}{\gamma^2 - 1}.$$

Clearly $\bar{v} \notin (-\infty, -1)$ for $\gamma > 1$. Hence, the maximum occurs at the boundary of the interval $(-\infty, -1)$. That is, when $v = v^* = -1$. So,

$$\lambda^*_{(-\infty,-1)} = \begin{cases} \infty & \gamma < 1 \\ -\gamma^2 & \gamma \geq 1 \end{cases}. \tag{5.9}$$

Similarly, combining (5.4) and (5.7), yields that

$$\lambda^*_{(1,\infty)} = \lambda^*_{(-\infty,-1)}. \tag{5.10}$$

Finally, combining (5.4) and (5.6),

$$\lambda^*_{[-1,1]} = 0, \tag{5.11}$$

for all gain $\gamma \geq 0$. So, combining (5.8) through (5.11),

$$\lambda_a = \begin{cases} \infty & \gamma < 1, \\ 0 & \gamma \geq 1, \end{cases} \tag{5.12}$$

♦

Equation (5.12) demonstrates that a unity gain deadzone has identical available power to that of the identity function. Note that is independent of the extend of the deadband.

Example 5.1.2 Consider a saturation nonlinearity with unity gain linear region $[-1, 1]$,

$$\sigma(v) = \begin{cases} 1 & v < -1, \\ -v & v \in [-1, 1], \\ -1 & v > 1. \end{cases} \tag{5.13}$$

Following the same procedure as in the preceding example,

$$\begin{aligned} \lambda^*_{(-\infty,-1)} = \lambda^*_{(1,\infty)} &= \sup_{v \in (-\infty,-1)} \left\{1 - \gamma^2 v^2\right\} \\ &= 1 - \gamma^2, \end{aligned} \tag{5.14}$$

for all $\gamma \geq 0$,

$$\begin{aligned} \lambda^*_{[-1,1]} &= \sup_{v \in [-1,1]} \left\{(1 - \gamma^2) v^2\right\} \\ &= \begin{cases} 1 - \gamma^2 & \gamma < 1, \\ 0 & \gamma \geq 1. \end{cases} \end{aligned} \tag{5.15}$$

Combining (5.8), (5.14), and (5.15),

$$\lambda_a = \begin{cases} 1 - \gamma^2 & \gamma < 1, \\ 0 & \gamma \geq 1. \end{cases} \tag{5.16}$$

♦

5.2 Linear Systems

Consider the class of linear systems described by the equations

$$\Sigma : \begin{cases} \dot{x} = Ax + Bv \\ z = Cx \end{cases}, \tag{5.17}$$

where $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times p}$, $C \in \mathbf{R}^{q \times n}$, with A a stable matrix.

Theorem 5.2.1 *Suppose system Σ given by (5.17) is $\mathcal{L}_2$-stable. Then,*

$$\lambda_a(x) = \begin{cases} \infty & \text{if } \gamma < \|\Sigma\|_{\mathcal{H}_\infty} \\ 0 & \text{if } \gamma \geq \|\Sigma\|_{\mathcal{H}_\infty} \end{cases} \tag{5.18}$$

Proof: $\underline{\gamma \geq \|\Sigma\|_{\mathcal{H}_\infty}}$: Any $\mathcal{L}_2$-stable linear system has $\mathcal{L}_2$-gain $\leq \gamma$ for all $\gamma \geq \|\Sigma\|_{\mathcal{H}_\infty}$. Hence, by Proposition 2.6.2, $\lambda_a(x) = 0$ for all $\gamma \geq \|\Sigma\|_{\mathcal{H}_\infty}$.

$\underline{\gamma < \|\Sigma\|_{\mathcal{H}_\infty}}$: Consider the application of sinusoidal inputs only. Since Σ is linear, the corresponding outputs must also be sinusoidal (plus some energy signal). So, for both input and output, the $\limsup_{T\to\infty}$ in the $\mathcal{FP}$-seminorm reduces to a $\lim_{T\to\infty}$, as in [45].

Now, define a sequence of frequencies, $\{\omega_n\}$ such that for input sinusoid $\hat{w}_n(t) = A\sin(\omega_n t)$, $A \in \mathbf{R}^n$, and corresponding output $\hat{z}_n(t)$, $\|\hat{z}_n\|_{\mathcal{FP}} \uparrow \|\Sigma\|_{\mathcal{H}_\infty}\|\hat{v}_n\|_{\mathcal{FP}}$ as $n \to \infty$. From [45], we know that such a sequence can always be found (since the $\mathcal{H}_\infty$-norm on the transfer matrix is induced by the $\mathcal{FP}$-seminorm defined therein). Now, each input $v_n = K\hat{v}_n$ is suboptimal, for all $K \in \mathbf{R}$ and all $n \geq 1$. So, using linearity,

$$\begin{aligned} \lambda_a^\gamma &\geq \sup_{n\geq 1, K\in\mathbf{R}} \left\{\|z_n\|^2_{\mathcal{FP}} - \gamma^2\|v_n\|^2_{\mathcal{FP}}\right\} \\ &= \sup_{n\geq 1, K\in\mathbf{R}} \left\{K^2\left[\|\hat{z}_n\|^2_{\mathcal{FP}} - \gamma^2\|\hat{v}_n\|^2_{\mathcal{FP}}\right]\right\}. \end{aligned}$$

But, $\gamma < \|\Sigma\|_{\mathcal{H}_\infty}$. Hence, for sufficiently large n,$\varepsilon_n > 0$, where

$$\varepsilon_n := \|\hat{z}_n\|^2_{\mathcal{FP}} - \gamma^2\|\hat{v}_n\|^2_{\mathcal{FP}}.$$

Thus,

$$\begin{aligned} \lambda_a(x) &\geq \sup_{n\geq 1, K\in\mathbf{R}} \left\{K^2\varepsilon_n\right\} \\ &= \infty, \end{aligned}$$

as required. ∎

This result only describes a sufficient condition (ie $\mathcal{L}_2$-stability) for which the available power is in the set $\{0, \infty\}$. To show that no intermediate values are possible, we must consider when Σ is not $\mathcal{L}_2$-stable.

Lack of $\mathcal{L}_2$-stability of Σ implies that the transfer matrix of Σ is either nonproper, or has RHP poles. In either case, it is easy to show that the available power is infinite. So, we have the following general result describing the available power for linear systems.

Theorem 5.2.2 *Linear system Σ can have finite available power iff Σ is $\mathcal{L}_2$-stable and the power gain factor γ is greater than or equal to $\|\Sigma\|_{\mathcal{H}_\infty}$. Furthermore, when the* *available power is finite, it is zero.*

Note that this result also shows that the available power is independent of the initial conditions. This can be proved directly, using superposition.

5.3 Affine Systems

5.3.1 General Results

Consider the class of affine nonlinear systems given by

$$\Sigma \ : \ \begin{cases} \dot{x}(t) & = & Ax(t) + Bv(t) \\ z(t) & = & Cx(t) + D \end{cases}, \tag{5.19}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times p}$, $C \in \mathbf{R}^{q\times n}$, and $D \in \mathbf{R}^{q}$. The following result provides a lower bound for the available power of a systems of the form (5.19).

Theorem 5.3.1 *Consider Σ affine as in (5.19), with A asymptotically stable. Then,*

$$\lambda_a \ \geq \ \begin{cases} D'D + D'CA^{-1}B[\gamma^2 I - S]^{-1}B'(A^{-1})'C'D & \text{if } \gamma^2 I - S > 0, \\ D'D & \text{if } \gamma^2 I - S = 0 \text{ and } \Sigma_\ell \\ & \text{has a zero at the origin,} \\ \infty & \text{otherwise,} \end{cases} \tag{5.20}$$

where $S = B'(A^{-1})'C'CA^{-1}B$,

$$\Sigma_\ell \ : \ \begin{cases} \dot{x}(t) & = & Ax(t) + Bv(t) \\ z_\ell(t) & = & Cx(t) \end{cases}. \tag{5.21}$$

Proof: Suppose that a constant disturbance $K \in \mathbf{R}^p$ is applied to the linear system Σ_ℓ (5.21). The Laplace Transform of the output is then

$$Z_\ell(s) \ = \ C(sI - A)^{-1}BK\frac{1}{s}.$$

Using the Final Value Theorem, the steady state output due to the constant disturbance K is the constant Z_ℓ (depending on K), where

$$\begin{aligned} Z_\ell &= \lim_{t\to\infty} \{z_\ell(t)\} \\ &= \lim_{s\to 0} \{sZ_\ell(s)\} \\ &= \lim_{s\to 0} \{C(sI - A)^{-1}BK\} \\ &= -CA^{-1}BK. \end{aligned}$$

Since Σ_ℓ is linear and A is asymptotically stable, the output can be written as

$$z_\ell(t) = \tilde{z}_\ell(t) + Z_\ell,$$

where $\tilde{z}_\ell$ is an energy signal. Hence, the output of the affine system Σ can be written as

$$z(t) = \tilde{z}_\ell(t) - Z \tag{5.22}$$

where $Z = -(Z_\ell + D)$. Hence,

$$\begin{aligned} |z(t)|^2 &= |\tilde{z}_\ell(t)|^2 - 2Z'\tilde{z}_\ell(t) + |Z|^2 \\ &\geq |\tilde{z}_\ell(t)|^2 - 2|Z||\tilde{z}_\ell(t)| + |Z|^2. \end{aligned}$$

A lower bound for the finite horizon value function $V(x,T)$ (2.33) can then be computed.

$$\begin{aligned} V(x,T) &= \sup_{v\in\mathcal{L}_2[0,T]} \left\{ \int_0^T \left[|z(s)|^2 - \gamma^2 |v(s)|^2 \right] \, ds \right\} \\ &\geq \sup_{K\in\mathbf{R}^p} \left\{ \int_0^T \left[|z(s)|^2 - \gamma^2 |K|^2 \right] \, ds \right\} \\ &\geq \sup_{K\in\mathbf{R}^p} \left\{ \int_0^T \left[|Z|^2 - \gamma^2 |K|^2 + |\tilde{z}_\ell(t)|^2 - 2|Z||\tilde{z}_\ell(t)| \right] \, ds \right\} \\ &= \sup_{K\in\mathbf{R}^p} \left\{ \left(|Z|^2 - \gamma^2 |K|^2 \right) T + \int_0^T \left[|\tilde{z}_\ell(t)|^2 - 2|Z||\tilde{z}_\ell(t)| \right] \, ds \right\} \end{aligned}$$

By definition of available power λ_a (2.84),

$$\begin{aligned} \lambda_a &= \limsup_{T\to\infty} \left\{ \frac{V(x,T)}{T} \right\} \\ &\geq \limsup_{T\to\infty} \sup_{K\in\mathbf{R}^p} \left\{ |Z|^2 - \gamma^2|K|^2 + \frac{1}{T}\int_0^T \left[|\tilde{z}_\ell(t)|^2 - 2|Z||\tilde{z}_\ell(t)| \right] \, ds \right\} \\ &\geq \sup_{K\in\mathbf{R}^p} \limsup_{T\to\infty} \left\{ |Z|^2 - \gamma^2|K|^2 + \frac{1}{T}\int_0^T \left[|\tilde{z}_\ell(t)|^2 - 2|Z||\tilde{z}_\ell(t)| \right] \, ds \right\} \\ &= \sup_{K\in\mathbf{R}^p} \left\{ |Z|^2 - \gamma^2|K|^2 + \limsup_{T\to\infty} \left\{ \frac{1}{T}\int_0^T \left[|\tilde{z}_\ell(t)|^2 - 2|Z||\tilde{z}_\ell(t)| \right] \, ds \right\} \right\} \\ &\geq \sup_{K\in\mathbf{R}^p} \left\{ |Z|^2 - \gamma^2|K|^2 + \limsup_{T\to\infty} \left\{ \frac{1}{T}\int_0^T |\tilde{z}_\ell(t)|^2 \, ds \right\} - \right. \\ &\qquad \left. 2\liminf_{T\to\infty} \left\{ \frac{1}{T}\int_0^T |Z||\tilde{z}_\ell(t)| \, ds \right\} \right\}. \end{aligned} \tag{5.23}$$

Hölder's inequality implies that

$$-\frac{1}{T}\int_0^T |Z||\tilde{z}_\ell(t)| \, ds \geq -\sqrt{\frac{1}{T^2}\int_0^T |Z|^2 \, ds \int_0^T |\tilde{z}_\ell(t)|^2 \, ds}$$

$$= -|Z|\sqrt{\frac{1}{T}\int_0^T |\tilde{z}_\ell(t)|^2\, ds}. \tag{5.24}$$

So, as $\tilde{z}_\ell(t)$ is an energy signal, (5.23) and (5.24) imply that

$$\begin{aligned}
\lambda_a &\geq \sup_{K\in\mathbf{R}^p}\left\{|Z|^2 - \gamma^2|K|^2 + \limsup_{T\to\infty}\left\{\frac{1}{T}\int_0^T |\tilde{z}_\ell(t)|^2\, ds\right\} - \right. \\
&\qquad\qquad \left. 2|Z|\liminf_{T\to\infty}\left\{\sqrt{\frac{1}{T}\int_0^T |\tilde{z}_\ell(t)|^2\, ds}\right\}\right\} \\
&= \sup_{K\in\mathbf{R}^p}\left\{|Z|^2 - \gamma^2|K|^2\right\}. \\
&= \sup_{K\in\mathbf{R}^p}\left\{|CA^{-1}BK - D|^2 - \gamma^2|K|^2\right\} \\
&= D'D + \sup_{K\in\mathbf{R}^p}\left\{K'\left(B'(A^{-1})'C'CA^{-1}B - \gamma^2 I\right)K - \left(2\, D'CA^{-1}B\right)K\right\} \\
&= D'D - \inf_{K\in\mathbf{R}^p} P(K),
\end{aligned}$$

where

$$P(K) = K'\left(\gamma^2 I - S\right)K + \left(2\, D'CA^{-1}B\right)K,$$

and $S = B'(A^{-1})'C'CA^{-1}B$. Note that $P(K)$ is quadratic in K. If $P(K)$ has negative concavity, then $\inf_{K\in\mathbf{R}^m} P(K) = -\infty$. Hence,

$$\gamma^2 I - S < 0 \;\Rightarrow\; \lambda_a \geq \infty. \tag{5.25}$$

Suppose that Σ_ℓ has a zero at the origin. Then, $CA^{-1}B = 0$. With $\gamma^2 I - S = 0$, $P(K)$ is then identically zero for all K. So,

$$\gamma^2 I - S = 0 \text{ and } \Sigma_\ell \text{ has a zero at the origin} \;\Rightarrow\; \lambda_a \geq D'D. \tag{5.26}$$

Alternatively, if Σ_ℓ has no zeros at the origin and $\gamma^2 I - S = 0$, then $P(K)$ is linear in K. Thus,

$$\gamma^2 I - S = 0 \text{ and } \Sigma_\ell \text{ has no zeros at the origin} \;\Rightarrow\; \lambda_a \geq \infty. \tag{5.27}$$

Finally, suppose that $\gamma^2 I - S > 0$. Then, $P(K)$ has positive concavity and hence a minimum. Differentiating $P(K)$ with respect to K,

$$\frac{\partial P(K)}{\partial K} = 2K'(\gamma^2 I - S) + 2\, D'CA^{-1}B.$$

Setting equal to zero and solving for the minimizing K,

$$K = K^* = -[\gamma^2 I - S]^{-1}B'(A^{-1})'C'D,$$

where we have used the fact that S is symmetric. Thus,

$$\inf_{K\in\mathbf{R}^p} P(K) = -D'CA^{-1}B[\gamma^2 I - S]^{-1}B'(A^{-1})'C'D.$$

So,

$$\gamma^2 I - S > 0 \;\Rightarrow\; \lambda_a \geq D'D + D'CA^{-1}B[\gamma^2 I - S]^{-1}B'(A^{-1})'C'D. \qquad (5.28)$$

Combining statements (5.25) through (5.28) completes the proof. ■

Theorem 5.3.2 *Consider an affine system* Σ *as in (5.19) with* A *asymptotically stable. Then, system* Σ *has* $\mathcal{FP}$*-gain* $\leq \gamma$ *for all* $\gamma > \|G\|_{\mathcal{H}_\infty}$*, where* $G(s) = C(sI - A)^{-1}B$*, with power bias*

$$\lambda = D'D + \frac{1}{4\gamma^2}QBB'Q',$$

where $Q = -2\; D'C(A + \frac{1}{\gamma^2}BB'P)^{-1}$*, and* P *is the stabilizing solution of the Algebraic Riccati Equation*

$$A'P + PA + \frac{1}{\gamma^2}PBB'P + C'C = 0.$$

Moreover, the worst case disturbance for this power bias is given by

$$v^*(t) = \frac{1}{\gamma^2}B'Px(t) + \frac{1}{2\gamma^2}B'Q'.$$

Proof: Suppose that (λ, V) is a viscosity solution pair of the PDE (2.179). Applying the completion of squares argument,

$$\nabla V(x)Ax + \frac{1}{4\gamma^2}\nabla V(x)BB'\nabla' V(x) + (Cx + D)'\,(Cx + D) = \lambda, \qquad (5.29)$$

where the worst case disturbance is $v^* = \frac{1}{2\gamma^2}B'\nabla' V(x)$. Suppose that V takes the form of a general quadratic function

$$V(x) = x'Px + Qx + R, \qquad (5.30)$$

where $P \in \mathbf{R}^{n\times n}$ is symmetric, $Q \in \mathbf{R}^{1\times n}$, and $R \in \mathbf{R}$. Substituting (5.30) in (5.29) and solving for the coefficients,

$$A'P + PA + \frac{1}{\gamma^2}PBB'P + C'C = 0 \qquad (5.31)$$

$$QA + \frac{1}{\gamma^2}QBB'P + 2D'C = 0 \qquad (5.32)$$

$$D'D + \frac{1}{4\gamma^2}QBB'Q = \lambda. \qquad (5.33)$$

Applying the Strict Bounded Real Lemma [35], $\gamma > \|G\|_{\mathcal{H}_\infty}$ implies that there always exists a nonnegative definite solution P of (5.31) such that $A^\times = A + \frac{1}{\gamma^2}BB'P$ has all eigenvalues in the open LHP. Hence, $A^\times$ is invertible and (5.32) implies that $Q = -2D'C(A^\times)^{-1}$ is uniquely defined. (5.33) then uniquely defines finite λ.

So, applying Theorem 2.7.3, system Σ has $\mathcal{FP}$-gain $\leq \gamma$. ■

Theorems 5.3.1 and 5.3.2 provide a lower bound and an upper bound (respectively) for the available power λ_a (2.84) of a multivariable affine system. We now show that the two bounds are in fact equal, thereby giving us an exact closed form expression for the available power. In order to do this, we begin by showing that the disturbances required to achieve the two bounds are identical.

Theorem 5.3.3 *The worst case disturbance v^* of Theorem 5.3.2 is equivalent to the worst case constant disturbance $K^* = -[\gamma^2 I - S]^{-1} B'(A^{-1})'C'D$ of Theorem 5.3.1, where $S = B'(A^{-1})'C'CA^{-1}B$.*

Proof: From Theorem 5.3.2, the optimal input is given by

$$w_* = \frac{1}{\gamma^2} B'P\xi^x(t) + \frac{1}{2\gamma^2} B'Q'.$$

Hence,

$$\dot{\xi}^x(t) = \underbrace{(A + \frac{1}{\gamma^2} BB'P)}_{A_{cl}} \xi^x(t) + \frac{1}{2\gamma^2} BB'Q'. \tag{5.34}$$

Since A_{cl} is a stability matrix, the steady state is

$$\xi^{ss} = -\frac{1}{2\gamma^2} A_{cl}^{-1} BB'Q',$$

and the steady state optimal input is thus

$$\begin{aligned} w_*^{ss} &= -\frac{1}{2\gamma^4} B'PA_{cl}^{-1}BB'Q' + \frac{1}{2\gamma^2} B'Q' \\ &= \left(\frac{1}{\gamma^4} B'PA_{cl}^{-1}BB' - \frac{1}{\gamma^2} B' \right) (A_{cl}^{-1})'C'D \\ &= \left[AA_{cl}^{-1} \left(\frac{1}{\gamma^4} BB'(A_{cl}^{-1})'PB - \frac{1}{\gamma^2} B \right) \right]' (A^{-1})'C'D. \end{aligned}$$

Multiplying on the left by the symmetric matrix $\gamma^2 I - S$,

$$(\gamma^2 I - S) w_*^{ss} = \left[\underbrace{AA_{cl}^{-1} \left(\frac{1}{\gamma^4} BB'(A_{cl}^{-1})'PB - \frac{1}{\gamma^2} B \right) (\gamma^2 I - S)}_{N} \right]' (A^{-1})'C'D.$$

Expanding the expression for N,

$$N = \frac{1}{\gamma^2} AA_{cl}^{-1} \left(\underbrace{BB'(A_{cl}^{-1})'PB}_{T_1} - \gamma^2 B - \underbrace{\frac{1}{\gamma^2} BB'(A_{cl}^{-1})'PBS}_{T_3} + BS \right).$$

Considering term T_1,

$$T_1 = BB'(A_{cl}^{-1})'PAA^{-1}B$$

$$= BB'(A'PA_{cl}^{-1})'A^{-1}B.$$

From (5.31),

$$A'P = -(PA_{cl} + C'C).$$

Hence,

$$\begin{aligned} T_1 &= -BB'(P + C'CA_{cl}^{-1})'A^{-1}B \\ &= -BB'PA^{-1}B - BB'(A_{cl}^{-1})'C'CA^{-1}B. \end{aligned}$$

Also,

$$\begin{aligned} T_3 &= \frac{1}{\gamma^2}BB'(A_{cl}^{-1})'PBS \\ &= BB'(A_{cl}^{-1})'(\frac{1}{\gamma^2}PBB')(A^{-1})'C'CA^{-1}B \\ &= BB'(A_{cl}^{-1})'A_{cl}'(A^{-1})'C'CA^{-1}B - BB'(A_{cl}^{-1})'A'(A^{-1})'C'CA^{-1}B \\ &= BB'(A^{-1})'C'CA^{-1}B - BB'(A_{cl}^{-1})'C'CA^{-1}B \\ &= BS - BB'(A_{cl}^{-1})'C'CA^{-1}B. \end{aligned}$$

Hence,

$$\begin{aligned} N &= \frac{1}{\gamma^2}AA_{cl}^{-1}\,(-BB'PA^{-1}B - BB'(A_{cl}^{-1})'C'CA^{-1}B - \gamma^2 B - BS + \\ &\qquad BB'(A_{cl}^{-1})'C'CA^{-1}B + BS) \\ &= -\frac{1}{\gamma^2}AA_{cl}^{-1}\,(\gamma^2 B + BB'PA^{-1}B) \\ &= -AA_{cl}^{-1}A_{cl}A^{-1}B \\ &= -B. \end{aligned}$$

Thus,

$$(\gamma^2 I - S)w_*^{ss} = -B'(A^{-1})'C'D$$

or

$$\begin{aligned} w_*^{ss} &= -(\gamma^2 I - S)^{-1}B'(A^{-1})'C'D \\ &= K_m, \end{aligned}$$

where K_m is as in Theorem 5.3.1. Since $w_* - w_*^{ss}$ is an energy signal, the available power is attained by application of input $w_*^{ss} = K_m$. ∎

Finally, to tie together completely the two bounds for λ_a^γ, we show that both apply for the same range of gain factors.

Lemma 5.3.4 *Suppose that $\gamma > \|G\|_{\mathcal{H}_\infty}$, where G is the transfer matrix of linear subsystem Σ_ℓ (5.21). Then, the matrix $\gamma^2 I - S$ is positive definite. That is,*

$$\gamma > \|G\|_{\mathcal{H}_\infty} \quad \Rightarrow \quad \gamma^2 I - S > 0.$$

Proof: Define

$$\begin{aligned} \sigma(j\omega) &= G^*(j\omega)G(j\omega) \\ &= B'\left((j\omega I - A)^{-1}\right)^* C'C(j\omega I - A)^{-1}B. \end{aligned}$$

Clearly, $\sigma(j0) = S$. By definition of the $\mathcal{H}_\infty$-norm,

$$\begin{aligned} \|G\|_{\mathcal{H}_\infty} &= \sup_{w\in\mathbf{R}} \max_i \left\{\sqrt{\alpha_i(j\omega)} : \sigma(j\omega)\nu = \alpha_i(j\omega)\nu\right\} \\ &\geq \max_i \left\{\sqrt{\alpha_i(j0)} : \sigma(j0)\nu = \alpha_i(j0)\nu\right\} \\ &= \max_i \{\sqrt{\alpha_i} : S\nu = \alpha_i\nu\} \end{aligned}$$

or

$$\|G\|^2_{\mathcal{H}_\infty} \;\geq\; \max_i \{\alpha_i : S\nu = \alpha_i\nu\}. \tag{5.35}$$

Hence, the eigenvalues of S (which are real since S is symmetric) cannot exceed the square of the $\mathcal{H}_\infty$-norm of the transfer matrix G. Now, denote the set of eigenvalues of $\gamma^2 I - S$ by $\{\beta_i : i \in [1, m]\}$. Then,

$$\left|(\gamma^2 I - S) - \beta_i I\right| = 0 \quad \Rightarrow \quad \left|(\gamma^2 - \beta_i)I - S\right| = 0,$$

which implies that $\gamma^2 - \beta_i$ is an eigenvalue of S. Hence, we can write that $\alpha_i = \gamma^2 - \beta_i$. So, applying (5.35),

$$\begin{aligned} \|G\|^2_{\mathcal{H}_\infty} &\geq \max_i \left\{\gamma^2 - \beta_i : (\gamma^2 I - S)\nu = \beta_i\nu\right\} \\ &= \gamma^2 - \min_i \left\{\beta_i : (\gamma^2 I - S)\nu = \beta_i\nu\right\}. \end{aligned}$$

But, by assumption, $\gamma^2 > \|G\|^2_{\mathcal{H}_\infty}$. Hence,

$$\min_i \left\{\beta_i : (\gamma^2 I - S)\nu = \beta_i\nu\right\} \;\geq\; \gamma^2 - \|G\|^2_{\mathcal{H}_\infty} > 0,$$

which states that all eigenvalues of $\gamma^2 I - S$ must be positive. Thus, the matrix $\gamma^2 I - S$ must be positive definite. ∎

So, having proved that the lower and upper bounds for the available power λ_a^γ apply over the same range of gain factors, we now proved the major result of this section.

Theorem 5.3.5 *Consider Σ as in (5.19), with A asymptotically stable. Then,*

$$\lambda_a^\gamma = \begin{cases} D'D + D'CA^{-1}B[\gamma^2 I - S]^{-1}B'(A^{-1})'C'D & \text{if } \gamma > \|G\|_{\mathcal{H}_\infty}, \\ D'D & \text{if } \gamma^2 I - S = 0 \text{ and } \Sigma_\ell \text{ has} \\ & \text{a zero at the origin,} \\ \infty & \text{if } \gamma < \|G\|_{\mathcal{H}_\infty}. \end{cases} \tag{5.36}$$

Example 5.3.6 For scalar affine systems, $S = \frac{b^2c^2}{a^2} = \|G\|^2_{\mathcal{H}_\infty}$, where G is the transfer matrix of the linear system corresponding to $d = 0$. Hence, by Theorem 5.3.5,

$$\begin{aligned} \lambda_a^\gamma &= d^2\left(1 + \frac{\frac{b^2c^2}{a^2}}{\gamma^2 - \frac{b^2c^2}{a^2}}\right) \\ &= \frac{\gamma^2 d^2}{\gamma^2 - \frac{b^2c^2}{a^2}} \end{aligned}$$

for all $\gamma > \frac{bc}{a}$. ♦

Example 5.3.7 Consider system Σ of the form of (5.19), with

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \\ -10 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Applying Theorem 5.3.5, the available power can be calculated for all $\gamma > \|G\|_{\mathcal{H}_\infty}$. A plot of the available power is shown in Figure 5.1. The dashed line is positioned at $\gamma = \|G\|_{\mathcal{H}_\infty}$. ♦

In the limit as $\gamma \to \infty$, the increasing penalty γ^2 on the disturbance energy implies that the available power λ_a^γ must tend to the available power of the correponding autonomous system ($v = 0$). Hence, $\lim_{\gamma\to\infty}\{\lambda_a^\gamma\} \downarrow D'D = 4$. Note also that as $\gamma \downarrow \|\Sigma_\ell\|$, the available power increases to a finite limit.

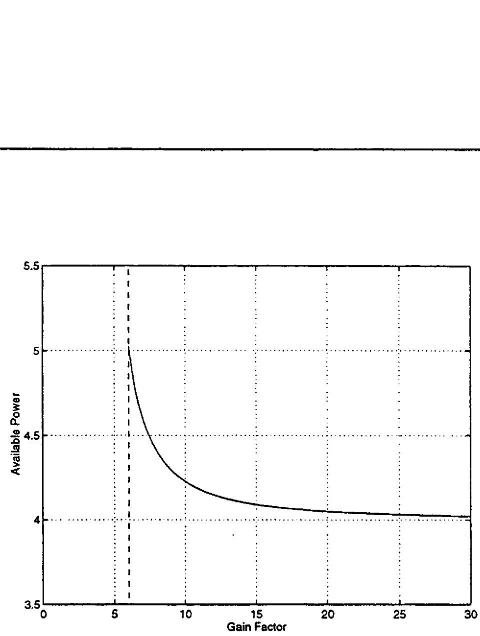


Figure 5.1: The Available Power λ_a^γ (2.84) for Example 5.3.7

5.3.2 Scalar Affine Systems without Disturbances

Consider the special case of affine systems without disturbances,

$$\Sigma \; : \; \begin{cases} \dot{x}(s) &= ax(s) \\ z(s) &= cx(s) + d \end{cases}$$

where $a < 0$. Applying Theorem 5.3.5, the available power is given by $\lambda_a = d^2$. Since the system is disturbance free, $x(s) = x\mathrm{e}^{as}$. So, the finite horizon value function $V(x,T)$ is given by (2.33),

$$\begin{aligned} V(x,T) &= \int_0^T |z(s)|^2 ds \\ &= \int_0^T \left[c^2x^2\mathrm{e}^{2as} + 2dcx\mathrm{e}^{as} + d^2\right] \, ds \\ &= \left[\left(\frac{c^2\mathrm{e}^{2as}}{2a}\right)x^2 + \left(\frac{2dc\mathrm{e}^{as}}{a}\right)x + d^2\right]_0^T \\ &= \left(\frac{c^2}{2a}(\mathrm{e}^{2aT}-1)\right)x^2 + \left(\frac{2dc}{a}(\mathrm{e}^{aT}-1)\right)x + d^2T \qquad (5.37) \end{aligned}$$

Applying the definition of available power (2.85) yields the expected result, $\lambda_a = d^2$.

The super available storage may be calculated from the definition (2.150).

$$\begin{aligned} V_a(x) &= \sup_{T\geq 0}\left\{\left(\frac{c^2x^2}{2a}\right)\mathrm{e}^{2aT} + \left(\frac{2dcx}{a}\right)\mathrm{e}^{aT} - \left(\frac{c^2x^2}{2a} + \frac{2dcx}{a}\right)\right\} \\ &= \rho(\mathrm{e}^{aT}), \end{aligned}$$

where $\rho(\cdot)$ is a quadratic function with negative concavity for $x \neq 0$. Hence, the supremum always exists. $\rho(\tau)$ has a maximum when $\tau = \tau^* = -\frac{2d}{cx}$. So, $V_a(x) = \rho(\tau^*)$ provided there exists a $T^* \geq 0$ such that $\mathrm{e}^{aT^*} = -\frac{2d}{cx} = \tau^*$. However, since $a < 0, T^* > 0$

implies that $0 \leq -\frac{2d}{cx} < 1$, or equivalently, that $x \in \left(-\infty, -\frac{2d}{c}\right)$. So, substituting τ^*,

$$\begin{aligned} V_a(x) &= \rho\left(-\frac{2d}{cx}\right) \\ &= \left(-\frac{c^2}{2a}\right)\left(x + \frac{2d}{c}\right)^2 \end{aligned}$$

if $x \in \left(-\infty, -\frac{2d}{c}\right)$. Next, consider $x \in \left[-\frac{2d}{c}, 0\right) \Rightarrow \tau^* \in [1, \infty)$. But, $\tau^* = e^{aT^*} \leq 1$, which implies that the supremum occurs at $\tau^* = 1$. Hence,

$$\begin{aligned} V_a(x) &= \rho(1) \\ &= 0 \end{aligned}$$

for all $x \in \left[-\frac{2d}{c}, 0\right)$. Note that since Σ has a stable equilibrium at $x = 0$, $V_a(0) = 0$. Finally, consider $x \in (0, \infty) \Rightarrow \tau^* \in (-\infty, 0]$. But, $\tau^* \geq 0$, which implies that $\tau^* = 0$. Hence,

$$\begin{aligned} V_a(x) &= \rho(0) \\ &= \left(-\frac{c^2}{2a}\right)x^2 + \left(-\frac{2dc}{a}\right)x \end{aligned}$$

for all $x \in (0, \infty)$. In summary, the super available storage is given by

$$V_a(x) = \begin{cases} \left(-\frac{c^2}{2a}\right)\left(x + \frac{2d}{c}\right)^2 & \text{if } x \in \left(-\infty, -\frac{2d}{c}\right), \\ 0 & \text{if } x \in \left[-\frac{2d}{c}, 0\right], \\ \left(-\frac{c^2}{2a}\right)x^2 + \left(-\frac{2dc}{a}\right)x & \text{if } x \in (0, \infty). \end{cases} \tag{5.38}$$

Note that this function is C^1. Taking the gradient and substituting in the Hamiltonian (see (2.163)),

$$H(x, \nabla_x V_a(x)) = \begin{cases} d^2 & \text{if } x \in \left(-\infty, -\frac{2d}{c}\right), \\ 0 & \text{if } x \in \left[-\frac{2d}{c}, 0\right], \\ d^2 & \text{if } x \in (0, \infty). \end{cases}$$

But, $d^2 = \lambda_a$. So, (λ_a, V_a) satisfies the PDE (2.179) when $V_a(x) > 0$, and the PDI (2.147) when $V_a(x) = 0$. Hence, (λ_a, V_a) satisfies the VI (2.163).

The infinite horizon available storage may be calculated directly from the definition (2.170). Applying (5.37) and noting that $a < 0$,

$$\begin{aligned} V_b(x) &= \limsup_{T \to \infty} \left\{ \left(\frac{c^2}{2a}(e^{2aT} - 1)\right)x^2 + \left(\frac{2dc}{a}(e^{aT} - 1)\right)x \right\} \\ &= \left(-\frac{c^2}{2a}\right)x^2 + \left(-\frac{2dc}{a}\right)x. \end{aligned}$$

Taking the gradient and substituting in the Hamiltonian (see (2.179)),

$$\begin{aligned} H(x, \nabla_x V_b(x)) &= \nabla_x V_b(x) a x + (cx+d)^2 \\ &= -\left(\left(\frac{c^2}{a}\right) x + \frac{2dc}{a}\right) ax + c^2x^2 + 2dcx + d^2 \\ &= d^2 = \lambda_a \end{aligned}$$

So, (λ_a, V_b) is a solution of the PDE (2.179).

5.3.3 Scalar Affine Systems with Disturbances

In Section 5.3.1, a general form for the available power and the corresponding stabilizing solution of the PDE (2.179) for multivariable affine systems was developed indirectly by constructing upper and lower bounds for the available power and proving that the bounds were in fact equal. For scalar affine systems, it is possible to solve the PDE (2.179) directly, without assuming any particular form for the solution V.

Consider the class of scalar affine systems with disturbances given by

$$\Sigma \ : \ \begin{cases} \dot{x} &= ax + bv, \\ z &= cx + d, \end{cases} \tag{5.39}$$

where $a < 0$, $b \neq 0$, and $c \neq 0$ (the system is stable, completely reachable, and observable). Applying completion of squares, PDE (2.179) may be written as

$$\frac{b^2}{4\gamma^2}\left[\nabla_x V(x)\right]^2 + ax\nabla_x V(x) + (cx+d)^2 - \lambda = 0 \tag{5.40}$$

Note that (5.40) is quadratic in the gradient term $\nabla_x V(x)$. The discriminant for this quadratic equation is

$$\begin{aligned} \Delta(x) &= a^2x^2 - \frac{b^2}{\gamma^2}\left[(cx+d)^2 - \lambda\right] \\ &= \frac{1}{\gamma^2}\left\{\left[\gamma^2 a^2 - b^2c^2\right]x^2 - 2b^2cdx + b^2\left[\lambda - d^2\right]\right\}. \end{aligned} \tag{5.41}$$

For a solution to the PDE (5.40) to exist, the discriminant $\Delta(x)$ must be nonnegative for all $x \in \mathbf{R}$. By inspection or (5.41), immediately we have that

$$\gamma > \left|\frac{bc}{a}\right| \tag{5.42}$$

With gain γ satisfying (5.42), $\Delta(x)$ has a global minimum at

$$x = x^* = \frac{b^2cd}{\gamma^2a^2 - b^2c^2} \tag{5.43}$$

Hence, for a solution of (5.40) to exist, the global minimum of $\Delta(x)$, $\Delta(x^*)$, must be nonnegative. That is,

$$\begin{aligned}\Delta(x^*) &= \frac{b^2}{\gamma^2}\left\{\lambda - \frac{\gamma^2 d^2}{\gamma^2 - \left|\frac{bc}{a}\right|^2}\right\} \\ &\geq 0\end{aligned}$$

Hence, the minimal power bias λ for which PDE (5.40) has a solution is

$$\lambda^* = \lambda_a = \frac{\gamma^2 d^2}{\gamma^2 - \left|\frac{bc}{a}\right|^2}. \tag{5.44}$$

Note that Theorem 2.10.15 ensures that $\lambda^* = \lambda_a$, since (as will be shown) there exists a corresponding stabilizing solution $V_-(x)$ of the PDE. Denoting the discriminant corresponding to $\lambda = \lambda^*$ by $\Delta^*(x)$,

$$\Delta^*(x) = \frac{\gamma^2 a^2 - b^2 c^2}{\gamma^2}\left[x - \frac{b^2 cd}{\gamma^2 a^2 - b^2 c^2}\right]^2. \tag{5.45}$$

So, the gradient of the solutions of the PDE (5.40) is given by

$$\begin{aligned}\nabla_x V(x) &= -\frac{2\gamma^2 ax}{b^2} \pm \frac{2\gamma^2}{b^2}\sqrt{\Delta^*(x)} &(5.46)\\ &= \frac{2\gamma^2|a|}{b^2}\left(1 \pm \sqrt{1-\mu^2}\right)x \mp \frac{2cd}{|a|}\left(\frac{1}{\sqrt{1-\mu^2}}\right), &(5.47)\end{aligned}$$

where $\mu = \left|\frac{bc}{\gamma a}\right|$. Integrating (5.47), the solutions of the PDE (5.40) are given by

$$V_{\pm}(x) = \frac{\gamma^2|a|}{b^2}\left(1 \pm \sqrt{1-\mu^2}\right)x^2 \mp \frac{2cd}{|a|}\left(\frac{1}{\sqrt{1-\mu^2}}\right)x + K, \tag{5.48}$$

where $K \in \mathbf{R}$ is any constant. Note that since $\mu < 1$ (5.42), both solutions are finite and bounded below.

The worst case disturbance for the system is (from the completion of squares) given by

$$\begin{aligned}v^* &= \frac{b}{2\gamma^2}\nabla_x V(x) \\ &= \frac{|a|}{b}\left(1 \pm \sqrt{1-\mu^2}\right)x \mp \frac{bcd}{\gamma^2|a|}\left(\frac{1}{\sqrt{1-\mu^2}}\right).\end{aligned}$$

Hence, the worst case dynamics are given by

$$\begin{aligned}\dot{x}^* &= -|a|x^* + bv^* \\ &= \pm|a|\left(\sqrt{1-\mu^2}\right)x^* \mp \frac{b^2 cd}{\gamma^2|a|}\left(\frac{1}{\sqrt{1-\mu^2}}\right). &(5.49)\end{aligned}$$

Consequently, in (5.48), the $-$ solution V_- is the stabilizing solution of (5.40), whilst the $+$ solution V_+ is the antistabilizing solution of (5.40).

From (5.49), it is possible to determine the stable and unstable equilibria corresponding to the stabilizing and antistabilizing solutions V_- and V_+ respectively. By setting $\dot{x}^* = 0$ in (5.49), we find that the two equilibria are identical, and are given by

$$x^* = \frac{b^2cd}{\gamma^2a^2 - b^2c^2},$$

which is precisely (5.43), the state which minimizes the discriminant. Since $\Delta^*(x^*) = 0$, (5.46) implies that $\nabla_x V_+(x^*) = \nabla_x V_-(x^*)$. Indeed, $V_+(x) - V_-(x)$ has a global minimum at x^*.

In order to compare the stabilizing and antistabilizing solutions of the stationary PDE (2.179) with the infinite horizon available storage $V_b(x)$ (2.169) and the infinite horizon required supply $V_{br}^f(\xi, x)$ (2.199) respectively, the corresponding finite horizon value functions $V(x,T)$ and $V_r^f(\xi, x, T)$ can be calculated numerically and the definitions (2.170) and (2.200) applied. One method of numerical calculation is to suppose that $V(x,T)$ (2.33) and $V_r^f(\xi, x, T)$ are of the form

$$V(x,T) = P_2(T)x^2 + P_1(T)x + P_0(T), \tag{5.50}$$

$$V_r^f(\xi, x, T) = Q_2(T)(x-\xi)^2 + Q_1(T)x + Q_0(T), \tag{5.51}$$

respectively. Substituting (5.50) in PDE (2.50) and equating coefficients yields the ODEs

$$\begin{aligned}
\dot{P}_2(T) &= 2aP_2(T) + \frac{b^2}{\gamma^2}P_2(T)^2 + c^2,\\
\dot{P}_1(T) &= \left[a + \frac{b^2}{\gamma^2}P_2(T)\right]P_1(T) + 2dc,\\
\dot{P}_0(T) &= d^2 + \frac{b^2}{4\gamma^2}P_1(T),
\end{aligned}$$

where $P_2(0) = 0$, $P_1(0) = 0$, and $P_0(0) = 0$, since $V(x,0) = 0$ for all $x \in \mathbf{R}^n$. Similarly, substituting (5.51) in PDE (2.198) and equating coefficients yields the ODEs

$$\begin{aligned}
\dot{Q}_2(T) &= -2aQ_2(T) - \frac{b^2}{\gamma^2}Q_2(T) - c^2,\\
\dot{Q}_1(T) &= -\left[a + \frac{b^2}{\gamma^2}Q_2(T)\right]Q_1(T) - 2cd - 2\xi\left[aQ_2(T) + c^2\right],\\
\dot{Q}_0(T) &= -d^2 - \frac{b^2}{4\gamma^2}Q_1(T)^2 + \frac{b^2}{\gamma^2}Q_1(T)Q_2(T)\xi + \xi^2\left[2aQ_2(T) + c^2\right].
\end{aligned}$$

State x is reachable from state ξ in zero time only if $x = \xi$. That is,

$$V_r^f(\xi, x, 0) = \begin{cases} 0 & x = \xi, \\ \infty & x \neq \xi. \end{cases}$$

This initial condition can be approximated numerically by choosing $Q_2(0) = R$, $Q_1(0) =$

0, and $Q_0(0) = 0$, where R is an arbitrarily large real number.

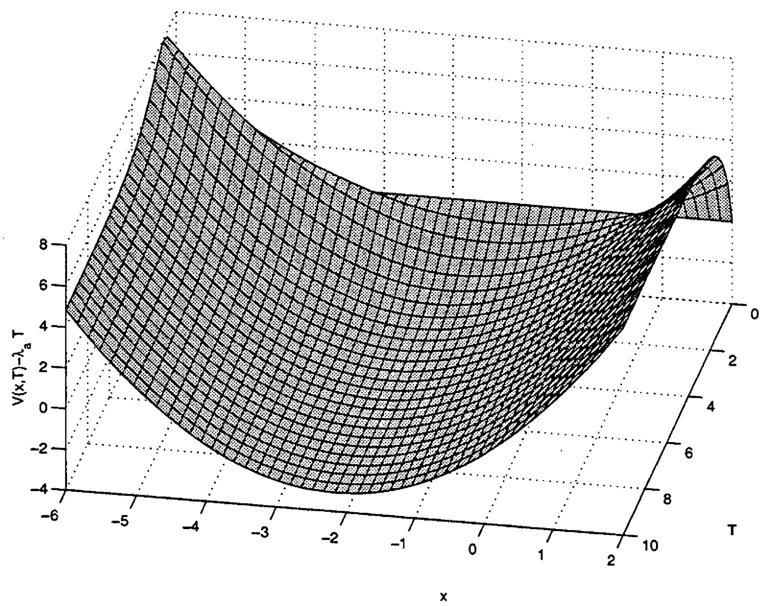


Figure 5.2: Numerical ODE Solution Approximation to $V(x,T) - \lambda_a T$ for Scalar Affine System (5.52)

Both of the above sets of ODEs may be solved numerically using conventional Runge-Kutta methods (using $\mathtt{MAPLE}_{\mathtt{TM}}$ for example). With

$$a = -1, \quad b = c = d = 1, \quad \gamma = 2, \tag{5.52}$$

$V(x,T) - \lambda_a T$ is illustrated in Figure 5.2. Similarly, $V_r^f(\xi, x, T) + \lambda_a T$ is illustrated in Figure 5.3, for $\xi = \text{argmin}_{x \in \mathbf{R}^n} \{V_b(x)\} \approx -2.15$. Note that V_r^f has been truncated above 80 for clarity. Using these approximations, $V_b(x)$ (2.169) and $V_{br}^f(\xi, x)$ (2.199) may be approximated by considering T large. The comparison between these approximations and the stabilizing solution $V_-(x)$ and antistabilizing solution $V_+(x)$ of the stationary PDE (5.40) is illustrated in Figure 5.4.

It is clear from Figure 5.4 that $V_b(x)$ is the stabilizing solution of the stationary PDE (5.40), whilst $V_{br}^f(\xi, x)$, $\xi = \text{argmin}_{x \in \mathbf{R}^n} \{V_b(x)\}$, is the antistabilizing solution. Figure 5.5 illustrates the relationship between $V_{br}^f(\xi, x)$ and $V_b(x)$, showing that $V_{br}^f(\xi, x) \geq \overline{V}_b(x)$ for $\xi = \text{argmin}_{x \in \mathbf{R}^n} \{V_b(x)\}$, as expected. Furthermore, from Figure 5.5, the W function (2.211) has a global minimum at $x^* = \frac{1}{3}$, which is precisely the equilibrium state for the worst case dynamics.

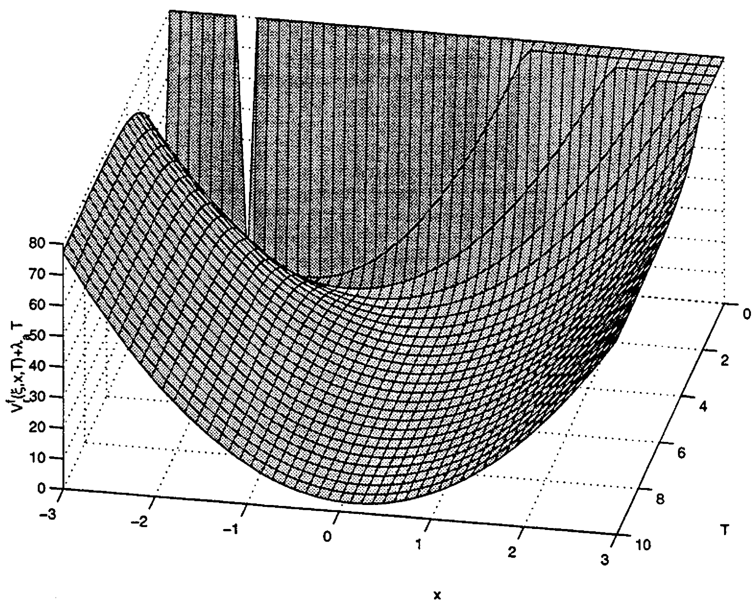


Figure 5.3: Numerical ODE Solution Approximation to $V_r^f(\xi, x, T) + \lambda_a T$ ($\xi \approx -2.15$) for Scalar Affine System (5.52)

Either finite differences or method of characteristics may also be applied to compute approximations for $V_b(x)$ and $V_{br}^f(\xi, x)$ for scalar affine systems. However, as the results are indistinguishable from the stationary PDE solutions shown in Figure 5.4, the results of the computations are omitted.

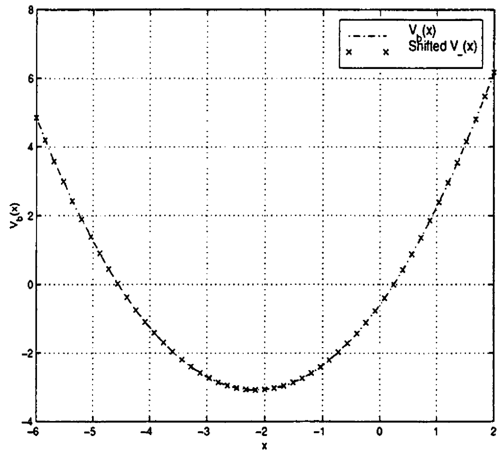
With the numerical approximation for the finite horizon value function $V(x,T) - \lambda_a T$ available (using the ODE method outlined), it is also possible to compute an approximation for the super available storage $V_a(x)$ (2.149) using equation (2.150). That is,

$$V_a(x) \;=\; \sup_{T\geq 0} \{V(x,T) - \lambda_a T\}\,.$$

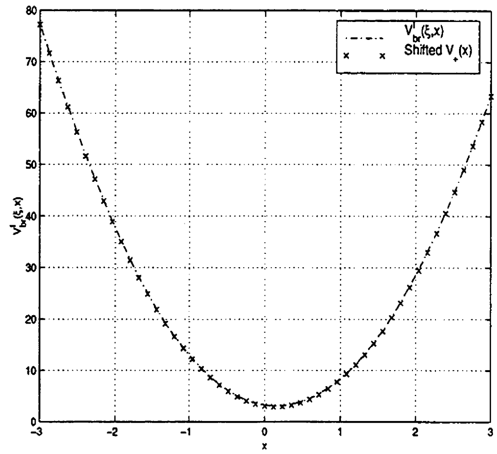
Hence, the super available storage may be interpreted as the "upper envelope" of $V(x,T) - \lambda_a T$ when viewed along the T direction in Figure 5.2. Exploiting the fact that (λ_a, V_a) satisfies the stationary VI (2.163) allows a finite difference approximation for $V_a(x)$ to be computed using the method of Section 4.7. With parameters (5.52) and state space / coordinate grids of

$$G_X \;=\; \{x \in \mathbf{R} : -4 \leq x \leq 1\} \cap (\mathbf{R})^{0.05},$$

$$G_V \;=\; \{v \in \mathbf{R} : -1 \leq v \leq 1\} \cap (\mathbf{R})^{0.01},$$



(a) The Infinite Horizon Available Storage $V_b(x)$ and the Stabilizing Solution $V_-(x)$



(b) The Infinite Horizon Fixed Initial State Required Supply $V_br^f(\xi, x)$ and the Anti-stabilizing Solution $V_+(x)$

Figure 5.4: Comparison between PDE solutions and Infinite Horizon Value Functions for Scalar Affine System (5.52)

a relative error of order 10^{-8} can be achieved after 500 value space iterations (with the available power $\lambda_a = \frac{4}{3}$, (5.44)). The resulting finite difference approximation is illustrated in Figure 5.6. Note that this figure also demonstrates the heirarchy of storage functions defined in Chapter 2. (The infinite horizon fixed initial state required supply shown is for initial state $\xi = \text{argmin}_{x\in\mathbf{R}^n}\{V_b(x)\}$.)

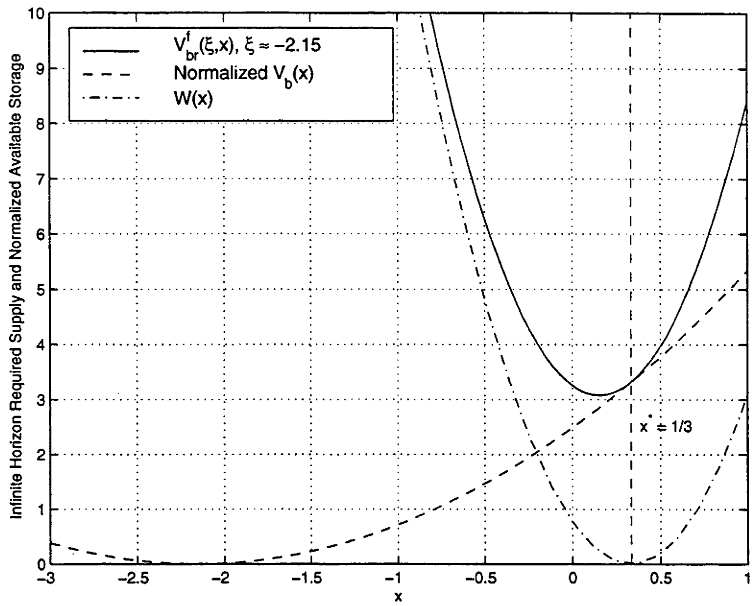


Figure 5.5: Comparison between the Infinite Horizon Fixed Initial State Required Supply $V^f_{br}(\xi, x)$, $\xi \approx -2.15$, and the Normalized Infinite Horizon Available Storage $\overline{V}_b(x)$ for Scalar Affine System (5.52)

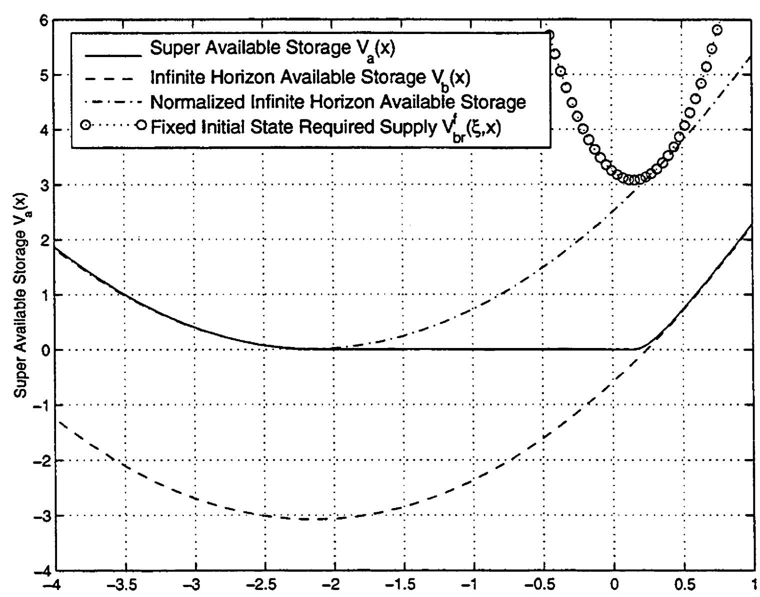


Figure 5.6: Super Available Storage $V_a(x)$ (2.149) for Scalar Affine System (5.52)

5.4 A Linear System with Saturating Output

Consider the class of scalar linear system with output saturation,

$$\Sigma : \begin{cases} \dot{x} &= ax + bv, \\ z &= h(x), \end{cases} \tag{5.53}$$

where

$$h(x) = \begin{cases} -c\varepsilon & x < -\varepsilon, \\ cx & |x| < \varepsilon, \\ c\varepsilon & x > \varepsilon. \end{cases}$$

For system (5.53), the stationary PDE (2.179) is given by

$$\sup_{v \in \mathbf{R}^p} \left\{ \nabla_x V(x) \cdot (ax + bv) + |h(x)|^2 - \gamma^2 |v|^2 \right\} = \lambda.$$

Equivalently, by completion of squares,

$$\frac{b^2}{4\gamma^2} \left(\nabla_x V(x)\right)^2 + ax \nabla_x V(x) + |h(x)|^2 - \lambda = 0, \tag{5.54}$$

which is clearly quadratic in $\nabla V(x)$. Hence, it is possible to solve for the gradient of the stabilizing solution,

$$\nabla_x V_-(x) = -\left(\frac{2\gamma^2 a}{b^2} x - \frac{2\gamma^2}{b^2} \right) \sqrt{a^2 x^2 - \frac{b^2}{\gamma^2} \left(|h(x)|^2 - \lambda \right)}, \tag{5.55}$$

which must hold for all $x \in \mathbf{R}$. This implies that the discriminant $\Delta(x) = a^2 x^2 - \frac{b^2}{\gamma^2} \left[|h(x)|^2 - \lambda \right]$ must be nonnegative for all $x \in \mathbf{R}$. Considering firstly $|x| < \varepsilon$, $\gamma \geq \left| \frac{bc}{a} \right|$, by definition $h(x) = cx$, and so

$$\Delta(x) = a^2 \left(1 - \mu^2\right) x^2 + \frac{b^2}{\gamma^2} \lambda,$$

where $\mu := \left| \frac{bc}{a\gamma} \right| \leq 1$. By inspection, $\Delta(x) \geq 0$ for all $x \geq 0$ for any $\lambda \geq 0$. Hence, $\lambda_a = 0$ for $|x| < \varepsilon$, $\gamma \geq \left| \frac{bc}{a} \right|$.

Alternatively, consider $\gamma < \left| \frac{bc}{a} \right| \Rightarrow 1 - \mu^2 < 0$. Then, $\Delta(x) \geq 0$ for all $|x| < \varepsilon$ if $\lambda \geq \frac{a^2 \varepsilon^2}{b^2} \left(\frac{b^2 c^2}{a^2} - \gamma^2 \right)$. So, for $|x| < \varepsilon$,

$$\lambda_a = \begin{cases} \frac{a^2 \varepsilon^2}{b^2} \left(\frac{b^2 c^2}{a^2} - \gamma^2 \right) & \gamma < \left| \frac{bc}{a} \right|, \\ 0 & \gamma \geq \left| \frac{bc}{a} \right|. \end{cases} \tag{5.56}$$

For $|x| > \varepsilon$, the output saturates at $\pm c\varepsilon$. In this case,

$$\Delta(x) = a^2 x^2 - \frac{b^2}{\gamma^2} \left(c^2 \varepsilon^2 - \lambda \right). \tag{5.57}$$

Again we find that $\Delta(x) \geq 0$ for all $|x| \geq \varepsilon$ provided that $\lambda \geq \frac{a^2 \varepsilon^2}{b^2} \left(\frac{b^2 c^2}{a^2} - \gamma^2 \right)$, for any $\gamma \geq 0$. Noting however that $\lambda_a \geq 0$ by definition, we get the same result as for $|x| < \varepsilon$.

That is, the available power for system Σ is given by

$$\lambda_a = \begin{cases} \frac{a^2\varepsilon^2}{b^2}\left(\frac{b^2c^2}{a^2} - \gamma^2\right) & \gamma < \left|\frac{bc}{a}\right|, \\ 0 & \gamma \geq \left|\frac{bc}{a}\right|. \end{cases} \tag{5.58}$$

Using the fact that any admissible power bias λ for the system is bounded below by the available power, (5.58) can be used to define the power gain for the system. Applying (2.22),

$$\gamma_\lambda^* = \begin{cases} \left|\frac{b}{a\varepsilon}\right| \sqrt{c^2\varepsilon^2 - \lambda} & \lambda \in [0, c^2\varepsilon^2), \\ 0 & \lambda \in [c^2\varepsilon^2, \infty). \end{cases}$$

The available power and the set of admissible power biases for the system are illustrated in Figure 5.7.

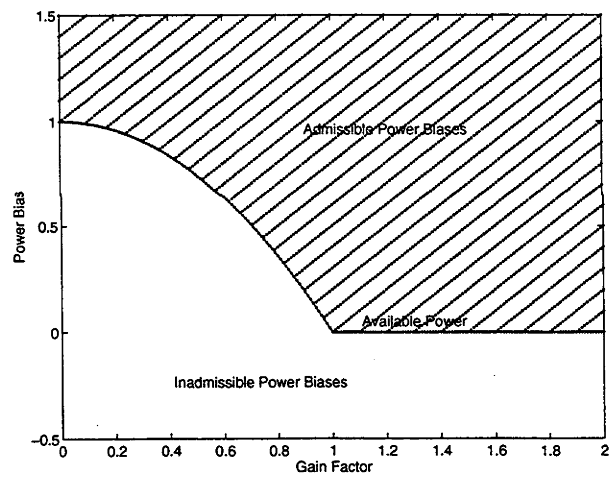


Figure 5.7: Admissible power biases for Σ with $a = -1$, $b = c = \varepsilon = 1$

Referring to the system class Σ (5.53), it is evident that as $\varepsilon \to \infty$, the output saturation is relaxed and the system becomes linear. Figure 5.8 demonstrates that as ε increases, the set of admissible power biases is restricted to that for a linear system.

Note that since the state space of system Σ is completely reachable, Theorem 2.6.5 states that the available power must be initial condition invariant. Since the state space of Σ is also locally uniformly reachable, Proposition 2.7.7 and Theorem 2.7.9 imply that any nonnegative function V satisfying the PDI (2.147) must be continuous. Using this information, the stabilizing solution of the stationary PDE (5.54) may be computed explicitly.

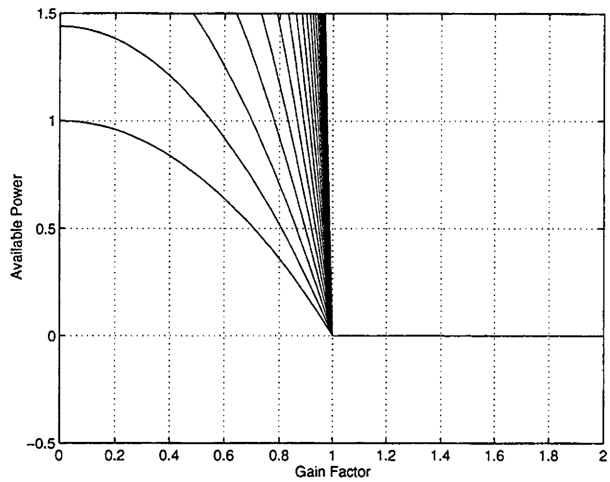


Figure 5.8: Available power for increasing ε with $a = -1$, $b = c = \varepsilon = 1$

Direct integration of (5.55) followed by application of Proposition 2.7.7 yields that for $\gamma \geq \left|\frac{bc}{a}\right|$,

$$V_-(x) = \begin{cases} -\frac{\gamma^2 a x^2}{b^2}\left(1 - \sqrt{1-\mu^2}\right) & |x| < \varepsilon, \\ -\frac{\gamma^2 a x^2}{b^2} + \frac{\gamma^2 a |x|}{b^2}\sqrt{x^2 - \mu^2\varepsilon^2} - \\ \qquad \frac{\gamma^2 a \mu^2 \varepsilon^2}{b^2}\log\left(\frac{|x| + \sqrt{x^2-\mu^2\varepsilon^2}}{\varepsilon + \varepsilon\sqrt{1-\mu^2}}\right) & |x| \geq \varepsilon, \end{cases} \tag{5.59}$$

where $\mu = \left|\frac{bc}{\gamma a}\right|$, and $V_-(x)$ is the stabilizing solution of (5.54). Similarly, for $\gamma < \left|\frac{bc}{a}\right|$,

$$V_-(x) = \begin{cases} -\frac{\gamma^2 a x^2}{b^2} - \frac{\gamma^2 a}{b^2}\sqrt{\mu^2 - 1}\cdot \\ \qquad \left(|x|\sqrt{\varepsilon^2 - x^2} + \varepsilon^2 \arcsin\left(\frac{|x|}{\varepsilon}\right)\right) & |x| < \varepsilon, \\ -\frac{\gamma^2 a x^2}{b^2} + \frac{\gamma^2 a |x|}{b^2}\sqrt{x^2 - \varepsilon^2} - \\ \frac{\gamma^2 a \varepsilon^2}{b^2}\log\left(\frac{|x| + \sqrt{x^2-\varepsilon^2}}{\varepsilon}\right) - \frac{\gamma^2 a \varepsilon^2 \pi}{2b^2}\sqrt{\mu^2-1} & |x| \geq \varepsilon. \end{cases} \tag{5.60}$$

An analogous computation can be performed for the antistabilizing solution.

Finally, since the system (5.53) is scalar, centered finite difference approximations for the infinite horizon available storage and infinite horizon fixed initial state required supply can easily be computed using a sequential computer. Two such computations are illustrated in the following two examples, highlighting the two cases for which (5.59) and (5.60) were derived.

Example 5.4.1 Consider a scalar linear system with saturating output of the form of (5.53), where

$$a = -1, \quad b = c = d = \varepsilon = 1, \quad \gamma = 0.5. \tag{5.61}$$

In order to apply the centered finite difference methods of Sections 4.3 and 4.8, define the state space and disturbance coordinate grids

$$\begin{aligned} G_X &= [-4.0, 4.0] \cap (\mathbf{R})^{0.05}, \\ G_V &= [-2.5, 2.5] \cap (\mathbf{R})^{0.01}, \end{aligned}$$

and

$$\begin{aligned} G_X^r &= G_X, \\ G_V^r &= [-5.0, 5.0] \cap (\mathbf{R})^{0.5}, \end{aligned}$$

for the infinite horizon available storage and required supply computations respectively. After 1000 iterations, the maximum relative error (over G_X) for the infinite horizon available storage approximation $V_{b,k}^{\delta}(x)$ has decreased to below the host computer (Sun SPARC) epsilon. Similarly, the maximum relative error for the infinite horizon required supply has decreased to order 10^{-12} after 1000 iterations. Figure 5.9 illustrates the evolution of the relative error for the two computations.

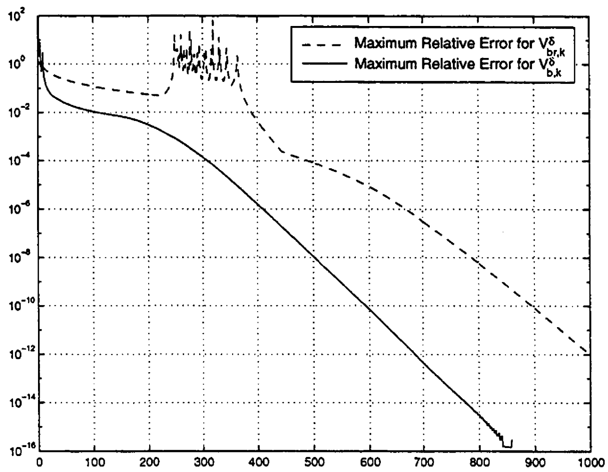


Figure 5.9: Convergence of Centered Finite Difference Approximations for $V_b(x)$ and $V_{br}^f(\xi, x)$, $\xi = 0$ (Example 5.4.1)

Figure 5.10 illustrates the comparison between the infinite horizon available storage approximation obtained and the stabilizing PDE solution given by (5.60) (both are normalized). Similarly, the worst case disturbance is illustrated in Figure 5.11. The evolution of the infinite horizon available storage is shown in Figure 5.12.

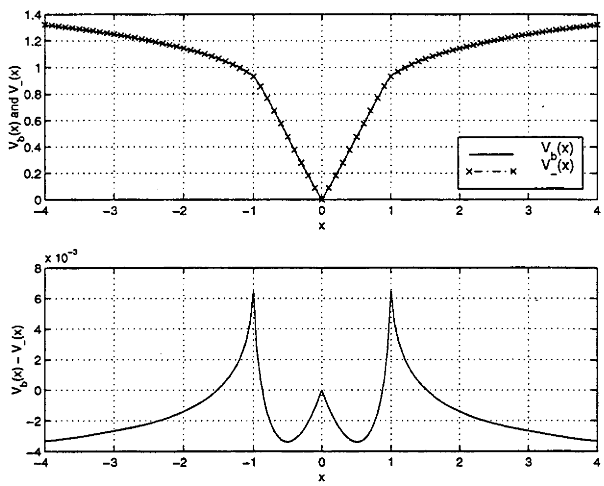


Figure 5.10: Comparison between Centered Finite Difference Approximation $V_b(x)$ and the Stabilizing Solution $V_-(x)$ (5.60) (Example 5.4.1)

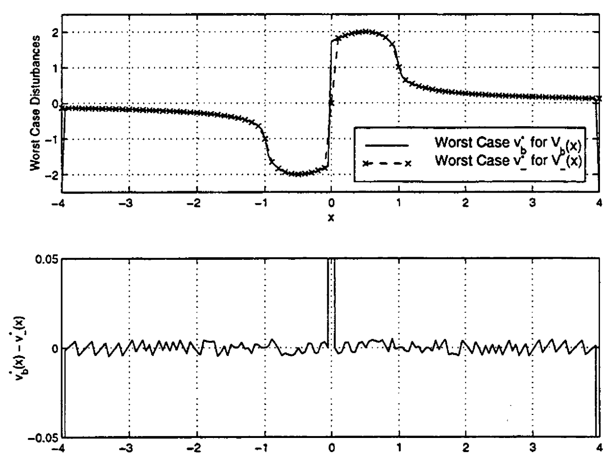


Figure 5.11: Comparison between Worst Case Disturbances for $V_b(x)$ and $V_-(x)$ (Example 5.4.1)

The centered finite difference methods of Sections 4.3, 4.4, and 4.5 also provide an approximation for the available power λ_a. By applying (5.58), the approximation for the available power may be compared with the expected value of $\lambda_a = 0.75$ for $\gamma = 0.5$. Figure 5.13 illustrates both the centered approximation and the average cost per unit time under the worst case disturbance. Clearly the approximations converge to the expected available power of 0.75.

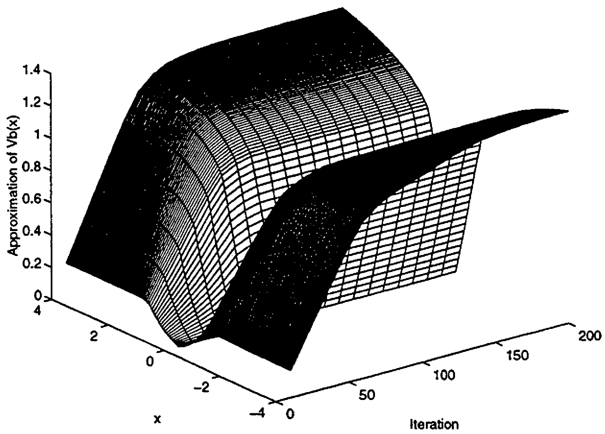


Figure 5.12: Evolution of the finite difference approximation of the infinite horizon available storage $V_b(x)$ (Example 5.4.1)

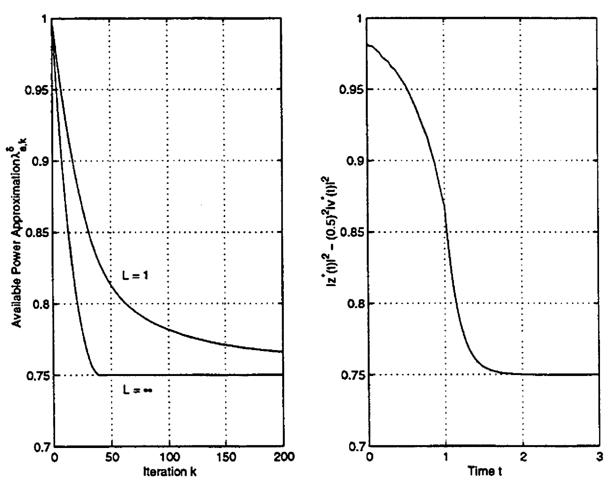


Figure 5.13: Available Power Approximations (Example 5.4.1)

The centered finite difference method of Section 4.8 may be applied to compute an approximation for the infinite horizon fixed initial state required supply $V_{br}^{f}(\xi, x)$ (2.199). From Section 2.14, the relevant choice of initial state is $\xi = \operatorname{argmin}_{x \in \mathbf{R}}\{V_b(x)\} = 0$. Figure 5.14 illustrates this approximation, along with the W function (2.211). Note that the minima of $W(x)$ occur at ± 1 approximately. Hence, Theorem 2.14.2 implies that any trajectories resulting from the forward time worst case disturbance for $V_b(x)$ should tend to ± 1. This is confirmed in Figure 5.15. Furthermore, $W(x)$ also decreases along reverse time worst case trajectories for $V_{br}^{f}(\xi, x)$. This is illustrated in Figure 5.16.

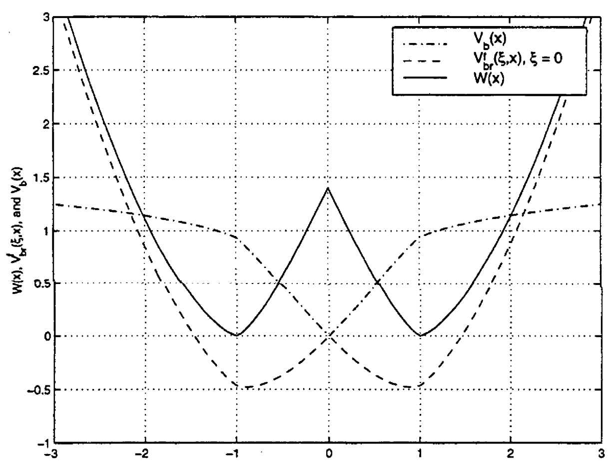


Figure 5.14: Approximations for $V_b(x)$, $V_{br}^{f}(\xi, x)$ ($\xi = 0$), and $W(x)$ (Example 5.4.1)

When the available power of a system is zero (see Example 5.4.2), the infinite horizon available storage $V_b(x)$ (2.169) and the super available storage $V_a(x)$ (2.149) coincide. However, for system (5.61), the available power is $\lambda_a = 0.75$. Hence, it is expected that $V_b(x)$ and $V_a(x)$ will differ. So, with the aim of comparing the two available storage functions, the finite difference method of Section 4.7 (with the available power approximation $\lambda_a^{\delta} = 0.75$ fixed) can be applied to compute an approximation for $V_a(x)$. Note that this method computes the solution of the stationary variational inequality (2.163).

As before, define the state space and disturbance coordinate grids

$$G_X = [-4.0, 4.0] \cap (\mathbf{R})^{0.05},$$

$$G_V = [-1.0, 1.0] \cap (\mathbf{R})^{0.01}. \tag{5.62}$$

Computing the approximation using 1000 value space iterations, Figure 5.20 illustrates the evolution of the maximum relative error (over G_X) for the super available storage approximation (for minimal and nonminimal interpolation times). As expected (see Chapter 4, the non-minimal interpolation method provides faster convergence.

A comparison of the super available storage $V_a(x)$ and the infinite horizon available storage $V_b(x)$ is illustrated in Figure 5.17. Notice that the functions coincide away from the zero set of $V_a(x)$. This follows directly from the variational inequality (2.163), since $V_a(x) > 0$ implies that $V_a(x)$ must satisfy the PDE (2.179). Notice also that the super available storage is differentiable. This is also noticeable in Figure 5.18, which compares the worst case disturbances for the two available storage functions.

Careful examination of the functions $V_a(x)$, $V_b(x)$ and $V_{br}^f(\xi, x)$ ($\xi = 0$) reveals that $V_a(x)$ is indeed differentiable at the boundary of the zero set. This is illustrated in Figure 5.19.

♦

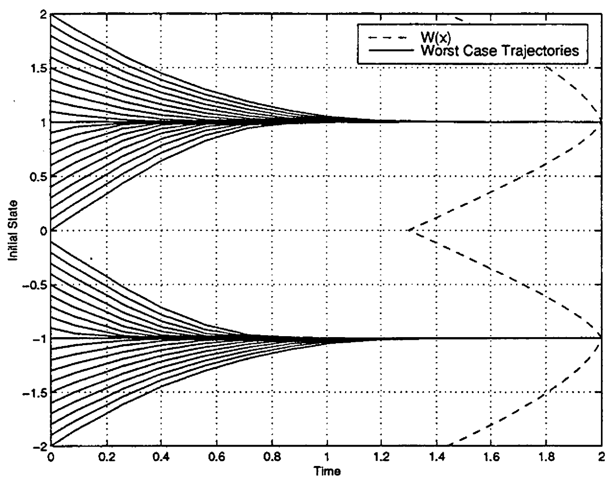


Figure 5.15: Worst Case Trajectories of $V_b(x)$ tend to the minimum of $W(x)$ (Example 5.4.1)

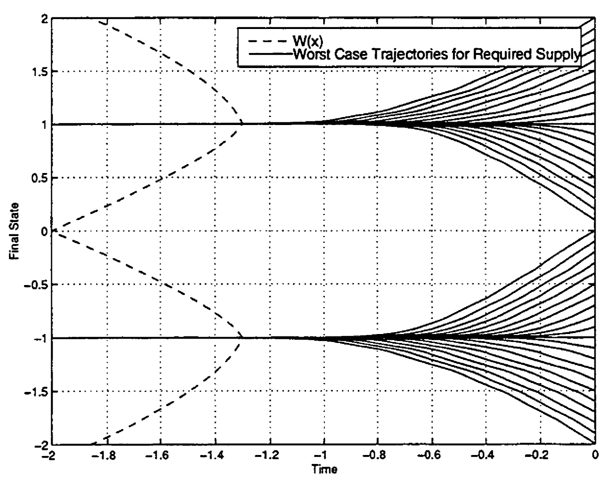


Figure 5.16: Worst Case Trajectories of $V_{br}^{f}(\xi, x)$, $\xi = 0$, tend to the minimum of $W(x)$ (Example 5.4.1)

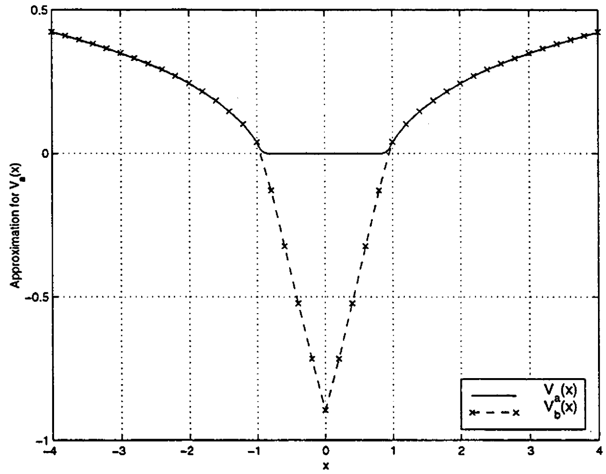


Figure 5.17: The super available storage $V_a(x)$ and the Infinite Horizon Available Storage $V_b(x)$ (Example 5.4.1)

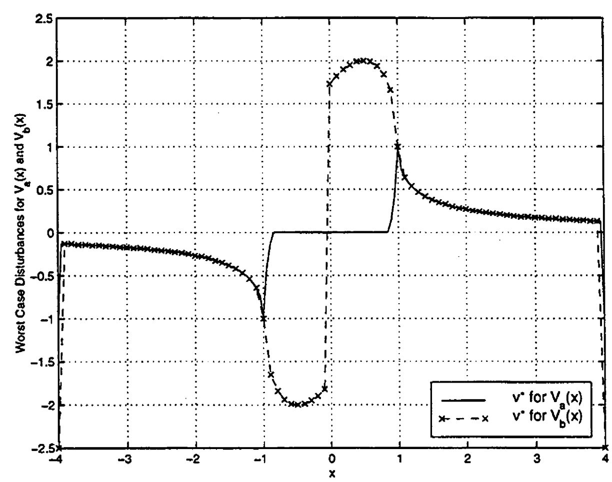


Figure 5.18: Approximate Worst Case Disturbances for $V_a(x)$ and $V_b(x)$ (Example 5.4.1)

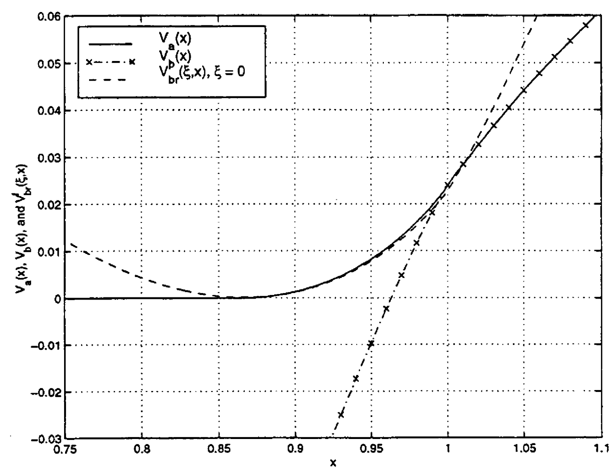


Figure 5.19: The super available storage $V_a(x)$ at the boundary of the level set (Example 5.4.1)

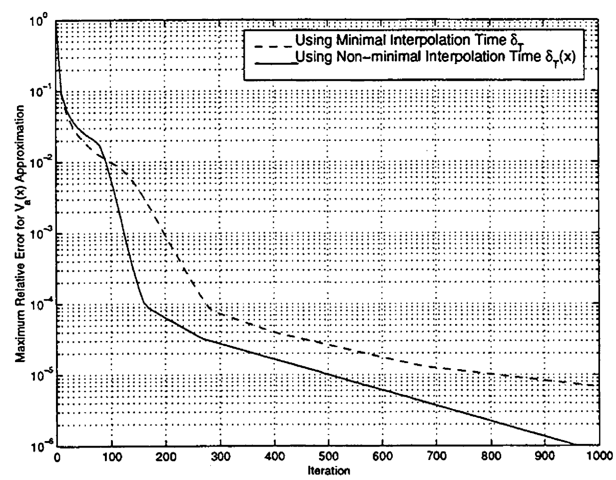


Figure 5.20: Convergence of Finite Difference Approximation for $V_a(x)$ (Example 5.4.1)

Example 5.4.2 Consider a scalar linear system with saturating output of the form of (5.53), where

$$a = -1, \quad b = c = d = \varepsilon = 1, \quad \gamma = 2. \tag{5.63}$$

As in Example 5.4.1, define the state space and disturbance coordinate grids

$$\begin{aligned} G_X &= [-4.00, 4.00] \cap (\mathbf{R})^{0.05}, \\ G_V &= [-0.15, 0.15] \cap (\mathbf{R})^{0.005}, \end{aligned}$$

and

$$\begin{aligned} G_X^r &= G_X, \\ G_V^r &= [-10.0, 10.0] \cap (\mathbf{R})^{0.5}, \end{aligned}$$

for the infinite horizon available storage and required supply computations respectively. After 1000 iterations, the relative error has decreased to order 10^{-8} for $V_{b,k}^{\delta}(x)$, and 10^{-5} for $V_{br,k}^{\delta}(\xi, x)$, $\xi = 0$. Figure 5.21 illustrates the evolution of these relative errors for the two computations.

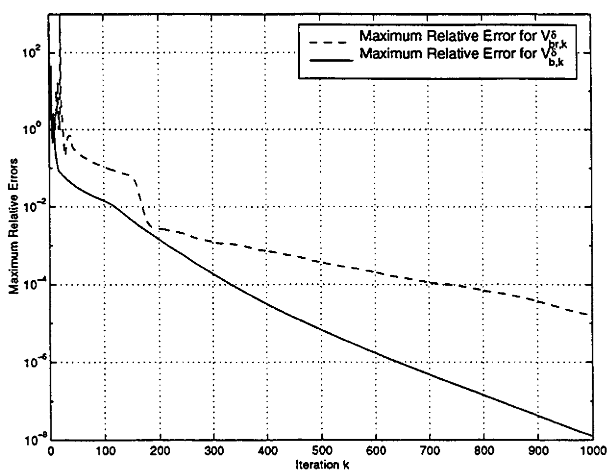


Figure 5.21: Convergence of Centered Finite Difference Approximations for $V_b(x)$ and $V_{br}^f(\xi, x)$, $\xi = 0$ (Example 5.4.2)

The comparison between the infinite horizon available storage approximation and the stabilizing PDE (5.54) solution is shown in Figure 5.22. Similarly, the comparison between worst case disturbances is illustrated in Figure 5.23. Notice that the error in the finite difference approximation for the worst case disturbance is less than the discretization used in the definition of G_V ($\delta_V = 0.005$).

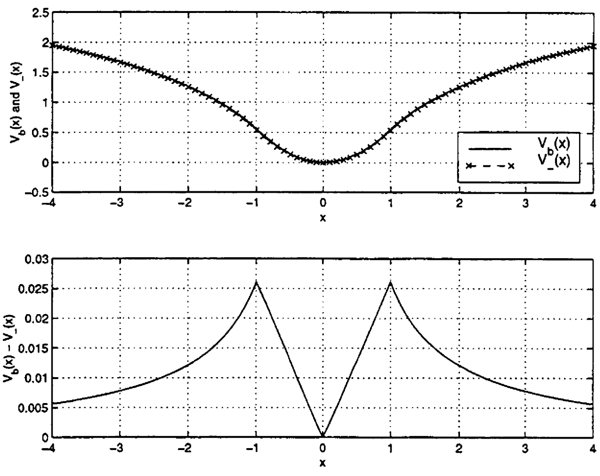


Figure 5.22: Comparison between Centered Finite Difference Approximation for $V_b(x)$ and the Stabilizing PDE Solution $V_-(x)$ (Example 5.4.2)

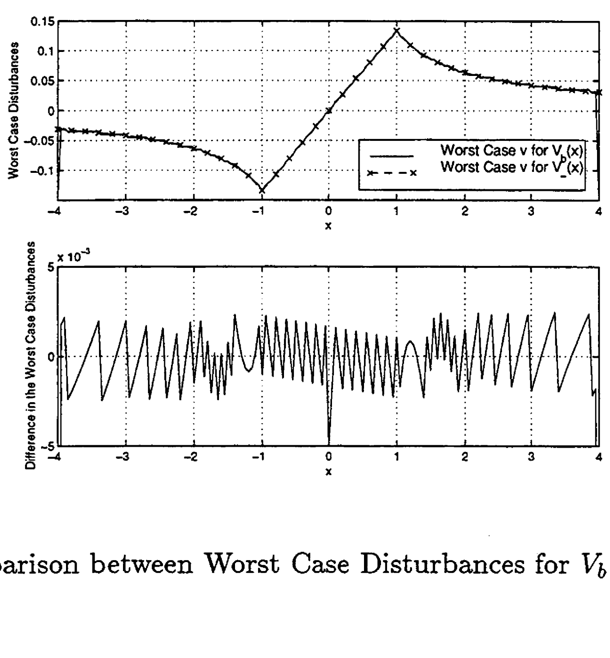


Figure 5.23: Comparison between Worst Case Disturbances for $V_b(x)$ and $V_-(x)$ (Example 5.4.2)

Figure 5.24 illustrates the finite difference approximations for the infinite horizon available storage $V_b(x)$ (2.169), the infinite horizon fixed initial state required supply $V_{br}^f(\xi,x)$ (2.199), and the $W(x)$ function (2.211). Note that $W(x)$ has a global minimum at the origin, implying that the trajectory corresponding to the worst case disturbance tends to zero. This is intuitively satisfying since (5.58) implies that the available power is zero for $\gamma = 2$.

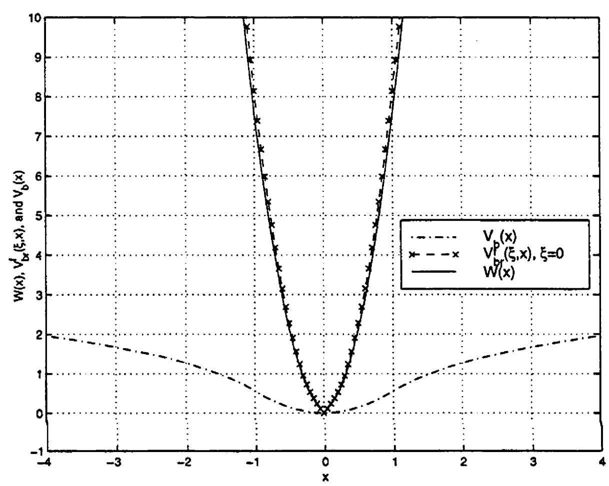


Figure 5.24: Approximations for $V_b(x)$, $V_{br}^f(\xi,x)$ $(\xi = 0)$, and $W(x)$ (Example 5.4.2)

Figures 5.25 and (5.26) demonstrate that the function $W(x)$ (2.211) decreases along the forward time worst case trajectory for $V_b(x)$ and the reverse time worst case trajectory for $V_{br}^f(\xi,x)$ $(\xi = 0)$. ♦

Comparison of the W functions for the $\gamma = 0.5$ (Example 5.4.1) case and the $\gamma = 2$ case (Figures 5.14 and 5.24) demonstrates a substantial difference in behaviour of the system under the influence of the worst case disturbance. In fact, since the number of minima of $W(x)$ is different for the two gains, a bifurcation must occurs for some intermediate $\gamma \in (0.5, 2)$. Refining this interval, Figure 5.27 demonstrates that the bifurcation must occur for $\gamma \in (0.95, 1.05)$. By computing $V_b(x)$ and $V_{br}^f(\xi,x)$ (using finite differences) for several immediate gains, it is possible to isolate the implied bifurcation, as shown in Figure 5.28.

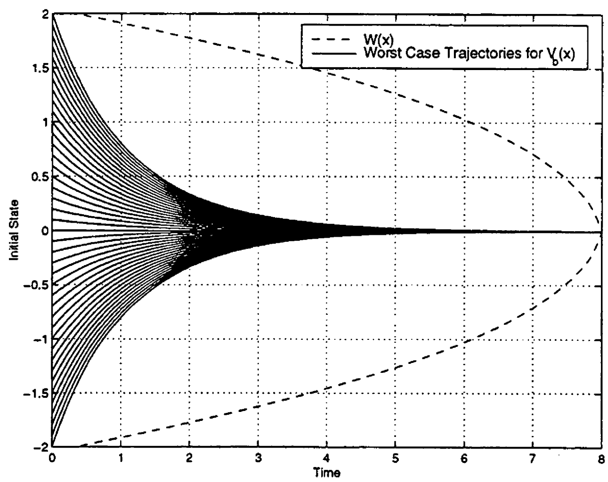


Figure 5.25: Worst Case Trajectories of $V_b(x)$ tend to the minimum of $W(x)$ (Example 5.4.2)

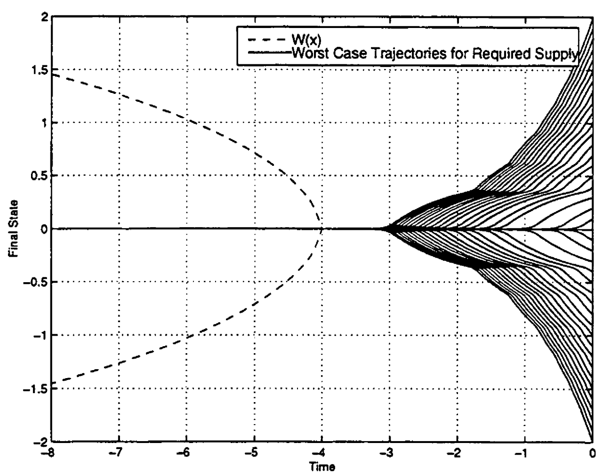


Figure 5.26: Worst Case Trajectories of $V^f_{br}(\xi, x)$ ($\xi = 0$) tend to the minimum of $W(x)$ (Example 5.4.2)

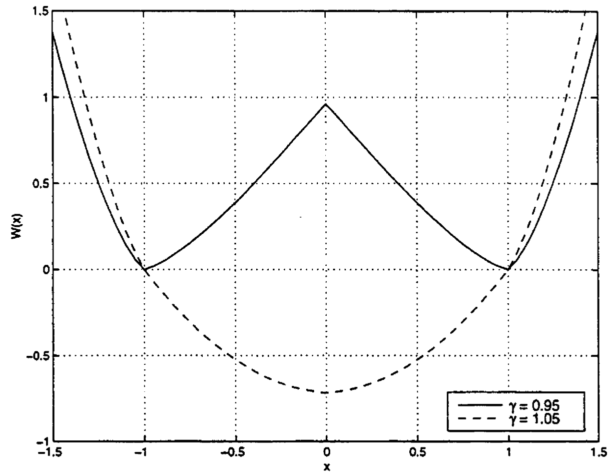


Figure 5.27: Bifurcation in the minimum of $W(x)$

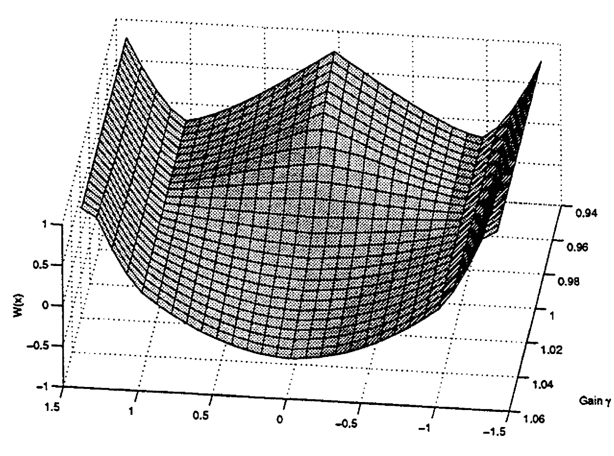


Figure 5.28: Bifurcation in the minimum of $W(x)$

5.5 A Cubic System without Disturbances

Consider the scalar unperturbed cubic system Σ given by

$$\Sigma \quad : \quad \begin{cases} \dot{x} &= -\frac{1}{2}x^3, \\ z &= x. \end{cases}$$

Integrating the state equation,

$$x(s) \quad = \quad \begin{cases} -\frac{1}{\sqrt{s+\frac{1}{x^2}}} & x < 0, \\ 0 & x = 0, \\ \frac{1}{\sqrt{s+\frac{1}{x^2}}} & x > 0. \end{cases}$$

As no disturbance enters the system, the finite horizon value function $V(x,T)$ (2.33) is given by

$$\begin{aligned} V(x,T) &= \int_0^T |z(s)|^2 \, ds \\ &= \int_0^T \frac{1}{s+\frac{1}{x^2}} \, ds \\ &= \log(1+x^2T). \end{aligned}$$

Hence, the available power λ_a (2.85) is given by

$$\begin{aligned} \lambda_a(x) &= \limsup_{T\to\infty} \left\{ \frac{V(x,T)}{T} \right\} \\ &= \limsup_{T\to\infty} \left\{ \frac{\log(1+x^2T)}{T} \right\} \\ &= 0 \end{aligned}$$

for all $x \in \mathbf{R}$. The super available storage V_a (2.150) is then given by

$$\begin{aligned} V_a(x) &= \sup_{T\geq 0} \{V(x,T) - \lambda_a T\} \\ &= \sup_{T\geq 0} \{\log(1+x^2T)\} \\ &= \infty \end{aligned} \tag{5.64}$$

for all $x \neq 0$. Hence, finiteness of the available power is not a sufficient condition for finiteness of the available storage. Note that the infinite horizon available storage V_b (2.170) is also infinite for $x \neq 0$.

Despite the lack of finiteness of the available storage, the system still exhibits the $\mathcal{FP}$-gain property. To illustrate this, consider a nonminimal power bias $\lambda > 0$. Writing

$$\hat{V}(x,T) \quad = \quad V(x,T) - \lambda T$$

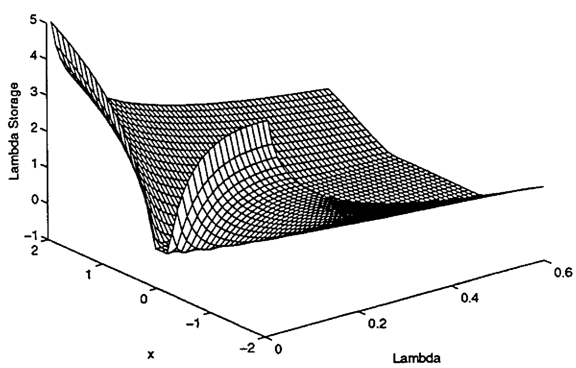


Figure 5.29: λ-storage for decreasing λ

$$= \log(1 + x^2 T) - \lambda T,$$

clearly $\hat{V}(x, T)$ has a maximum (over T) when

$$T = T^* = \begin{cases} \frac{x^2-\lambda}{x^2\lambda} & |x| > \sqrt{\lambda}, \\ 0 & |x| \le \sqrt{\lambda}. \end{cases}$$

Hence, the super λ-storage $V_{a\lambda}$ (2.164) is

$$V_\lambda(x) = \begin{cases} \log\left(\frac{x^2}{\lambda}\right) - \frac{x^2-\lambda}{x^2} & |x| > \sqrt{\lambda}, \\ 0 & |x| \le \sqrt{\lambda}, \end{cases}$$

which is finite (and C^1) for every $\lambda > 0$, as illustrated in Figure 5.29.

5.6 A Locally Cubic System with Disturbances

Consider the system

$$\Sigma \; : \; \begin{cases} \dot{x} = a(x) + v, \\ z = x, \end{cases} \tag{5.65}$$

where the drift term $a(x)$ is cubic inside a ball and affine elsewhere,

$$a(x) = \begin{cases} -2x - 2, \\ -x(x-1)(x+1), \\ -2x + 2. \end{cases} \tag{5.66}$$

The drift term (5.66) is illustrated in Figure 5.30.

By inspection, the system has stable equilibria at ± 1 and an unstable equilibrium

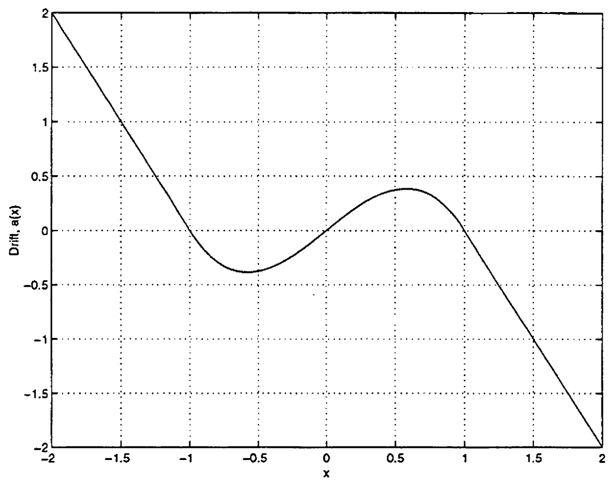


Figure 5.30: Drift $a(x)$ (5.66) for the Locally Cubic System (5.65)

at 0. Hence, in the absence of disturbances, the trajectory will tend to ± 1 for all initial $x \neq 0$. Since the system is completely reachable, Theorem 2.6.5 implies that the available power must be independent of the initial state. Furthermore, Remark 2.6.3 implies that $\lambda_a \geq 1$, due to the stable equilibria at ± 1. Note also that assumptions (A10) and (A12) hold for this system. Examination of Figure 5.31 indicates that assumption (A7) also holds. Hence, by application of Theorem 2.4.8, the system must have $\mathcal{FP}$-gain $\leq \sqrt{2}$, with a power bias of 6. By Theorem 2.6.4, this implies that $\lambda_a^\gamma \leq 6$ for all $\gamma \geq \sqrt{2}$.

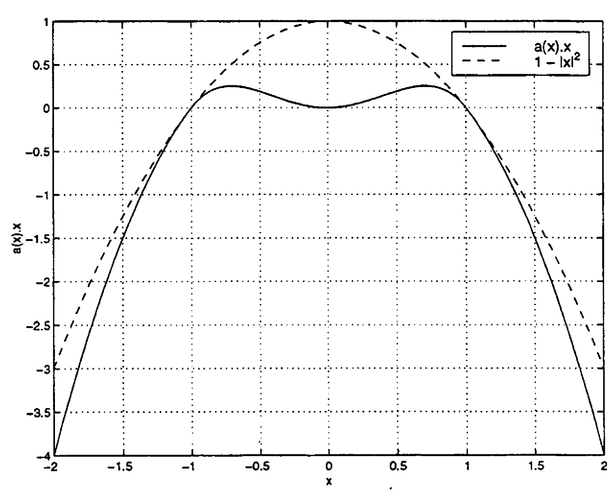


Figure 5.31: $a(x) \cdot x$ for the Locally Cubic System (5.65)

In order to calculate explicitly the available power for this system, consider the stationary PDE (2.179),

$$\frac{1}{4\gamma^2}\left[\nabla_x V(x)\right]^2 + a(x)\nabla_x V(x) + x^2 - \lambda \;=\; 0. \qquad (5.67)$$

The discriminant of this quadratic is then

$$\Delta(x) \;=\; \frac{1}{\gamma^2}\left\{\gamma^2\left[a(x)\right]^2 - x^2 + \lambda\right\},$$

which must be nonnegative for all $x \in \mathbf{R}$ for a solution of the PDE (5.67) to exist. Hence, any power bias λ for the system must satisfy

$$\lambda \;\geq\; \sup_{x\in\mathbf{R}}\left\{x^2 - \gamma^2\left[a(x)\right]^2\right\} =: \lambda^* \qquad (5.68)$$

Since the drift term (5.66) is piecewise, it is necessary to calculate λ^* for each subinterval of the state space R. Applying (5.68),

$$\lambda^*_{(-\infty,-1)} \;=\; \sup_{x<-1}\left\{\left(1-4\gamma^2\right)x^2 - 8\gamma^2 x - 4\gamma^2\right\}.$$

Since $\lambda_a < \infty$, the supremum must be finite. Hence, immediately it follows that $1 - 4\gamma^2 < 0$, or $\gamma > \frac{1}{2}$. Then, the quadratic argument has a global minimum at $x = x^* = \frac{4\gamma^2}{1-4\gamma^2}$, which from preceding gain constraint, must be less than -1. Hence, the supremum is achieved in the interval $(-\infty, -1)$, and

$$\lambda^*_{(-\infty,-1)} \;=\; \frac{4\gamma^2}{4\gamma^2 - 1}$$

for all $\gamma > \frac{1}{2}$. Similarly for $x > 1$,

$$\lambda^*_{(1,\infty)} \;=\; \frac{4\gamma^2}{4\gamma^2 - 1}$$

for all $\gamma > \frac{1}{2}$, with $x^* = -\frac{4\gamma^2}{1-4\gamma^2}$. Note that in both cases, $\lambda^* > 1$ for all gains. Finally, from (5.68),

$$\begin{aligned}
\lambda^*_{[-1,1]} \;&\leq\; \max_{x\in[-1,1]}\left\{x^2\right\} - \gamma^2 \min_{x\in[-1,1]}\left\{\left[a(x)\right]^2\right\} \\
&=\; 1 - \gamma^2\cdot 0 \\
&=\; 1
\end{aligned}$$

Theorem 2.10.15 state that the power bias satisfying the PDE (2.179) must be the available power λ_a, and hence unique. Furthermore, from completely reachability, λ_a must be independent of the initial state. Hence, the available power must be the minimal power bias which satisfies $\lambda \geq \lambda^*$ in all three intervals. That is,

$$\lambda_a \;=\; \max\left\{\lambda^*_{(-\infty,-1)}, \lambda^*_{[-1,1]}, \lambda^*_{(1,\infty)}\right\}$$

$$= \frac{4\gamma^2}{4\gamma^2 - 1} \tag{5.69}$$

for all $\gamma > \frac{1}{2}$. Note that for $\gamma = \sqrt{2}$, $\lambda_a = \frac{8}{7}$, which is considerably less than the power bias (= 6) obtained by application of Theorem 2.4.8 (note that since $c(x) = x^2$ for this example, we may apply (2.31), which yields a much reduced power bias of 2). The available power as a function of the gain is illustrated in Figure 5.32.

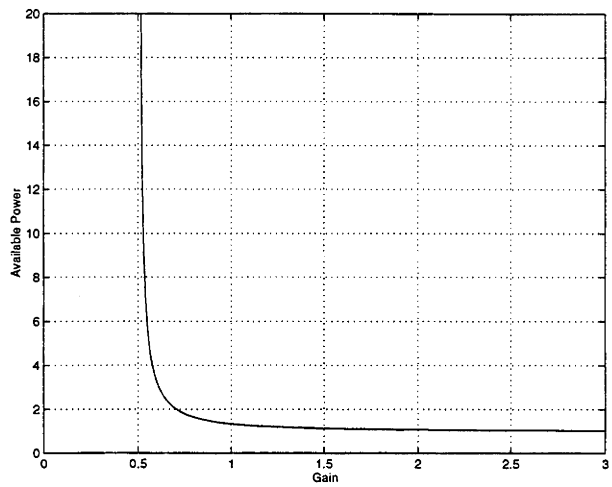


Figure 5.32: The Available Power λ_a for the Locally Cubic System (5.65)

As the available power λ_a is a differentiable function of the gain γ for gains $> \frac{1}{2}$, Theorem 2.6.11 may be applied to determine the power of the worst case disturbance. (2.126) yields that

$$\|v^*\|_{\mathcal{FP}} = \frac{2}{4\gamma^2 - 1},$$

which is clearly non zero. This can be verified by calculating the worst case disturbance using the stabilizing solution of the PDE (5.67). As in preceding examples, the gradient of the stabilizing solution can be found to be

$$\nabla_x V_-(x) = \begin{cases} -2\gamma^2 a(x) - 2\gamma^2\sqrt{\Delta^*(x)} & x \in (-\bar{x}, 0) \cup [\bar{x}, \infty), \\ -2\gamma^2 a(x) + 2\gamma^2\sqrt{\Delta^*(x)} & x \in (-\infty, -\bar{x}]) \cup (0, \bar{x}), \end{cases}$$

where $\Delta^*(\bar{x}) := \frac{1}{\gamma^2}\left\{\gamma^2 [a(\bar{x})]^2 - \bar{x}^2 + \lambda_a\right\} = 0$. Note that at $x = 0$, the gradient of the stabilizing solution is not well defined. As we will see later, this corresponds to a point of nonsmoothness in the stabilizing solution. The worst case disturbance (2.188) is

$$v^*(x) = \frac{1}{2\gamma^2}\nabla_x V_-(x)$$

So, the worst case dynamics are given by

$$\dot{x}^* = \begin{cases} -\sqrt{\Delta^*(x)} & x \in (-\bar{x}, 0) \cup [\bar{x}, \infty), \\ +\sqrt{\Delta^*(x)} & x \in (-\infty, -\bar{x}]) \cup (0, \bar{x}), \end{cases}$$

which has stable equilibria at $\pm\bar{x} = \pm\frac{4\gamma^2}{4\gamma^2-1}$. So,

$$\begin{aligned} \lim_{s\to\infty} \{v^*(x^*(s))\} &= -a(x^*) \\ &= \begin{cases} +\frac{2}{4\gamma^2-1} & x^*(0) < 0, \\ -\frac{2}{4\gamma^2-1} & x^*(0) > 0. \end{cases} \end{aligned}$$

Hence, $\|v^*\|_{\mathcal{FP}} = \frac{2}{4\gamma^2-1}$ as before.

By applying finite differences [9], [29], it is possible to compute approximations for V_b, V_{br}^f, and hence W, directly from the definitions (2.169), (2.199), and (2.211). This is accomplished by computing approximations for $V(x,T)$ (2.33) and $V_r^f(\xi, x, T)$ (2.194) using their respective nonstationary PDEs, see Chapter 4. With $\gamma = \frac{\sqrt{5}}{4}$, we expect $\lambda_a = \frac{5}{4}$, and equilibria under the worst case dynamics of (coincidently) $x^* = \pm\frac{5}{4}$. The results are shown in Figure 5.33, where $\xi = 0 = \operatorname{argmin}_x \{V_b(x)\}$.

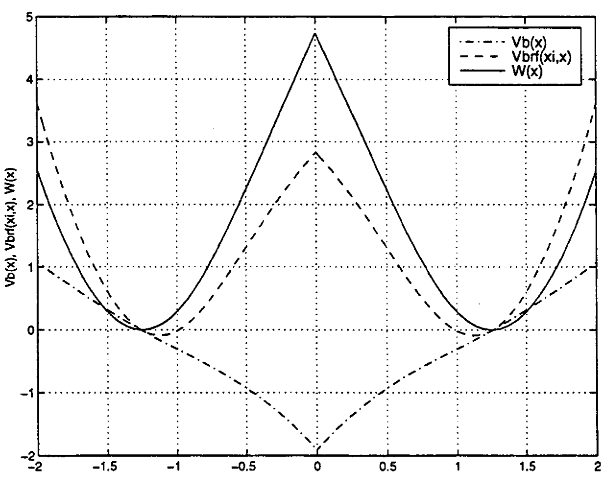


Figure 5.33: Functions $V_b(x)$, $V_{br}^f(\xi, x)$, and $W(x)$ for Locally Cubic System (5.65)

In order to check that (λ_a, V_b) is a solution pair of the PDE (2.179), we compute the Hamiltonian (2.51) using the computed V_b of Figure 5.33, see Figure 5.36. Note that the expected value is $H(x, \nabla_x V_b(x)) = \lambda_a = \frac{5}{4}$ for all $x \in \mathbf{R}^n$.

Finally, we demonstrate in Figure 5.34 that the worst case trajectories tend to the minimum points of $W(x)$ when the system is simulated in the presence of the worst case disturbance for $V_b(x)$. Similarly, $W(x)$ decreases along the worst case trajectory for $V_{br}^f(\xi, x)$, $\xi = 0$, as shown in Figure 5.35.

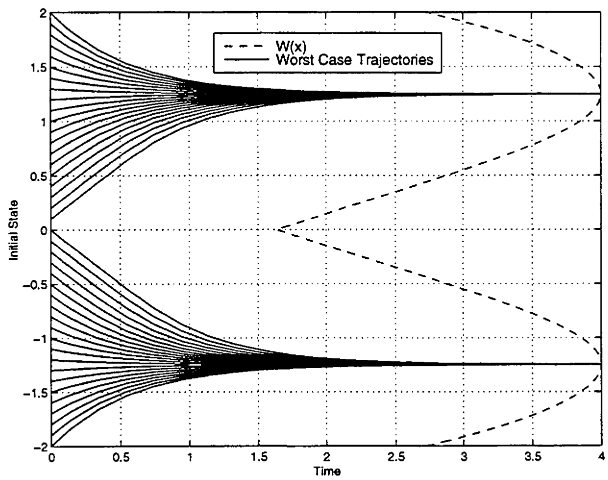


Figure 5.34: Worst Case Trajectories for $V_b(x)$ tend to the minimum of $W(x)$ for Locally Cubic System (5.65)

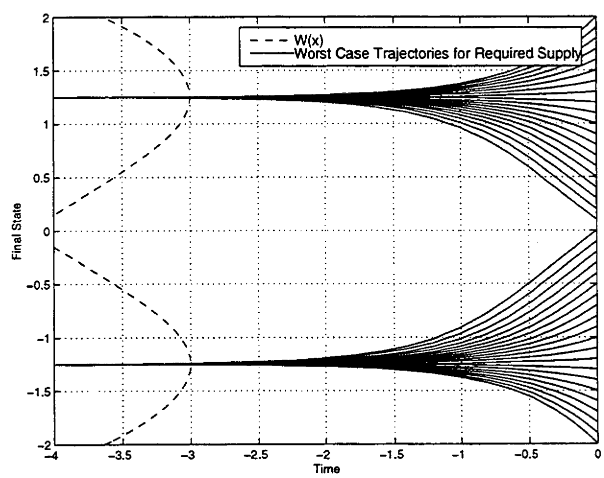


Figure 5.35: Worst Case Trajectories for $V_{br}^f(\xi, x)$ $(\xi = 0)$, tend to the minimum of $W(x)$ for Locally Cubic System (5.65)

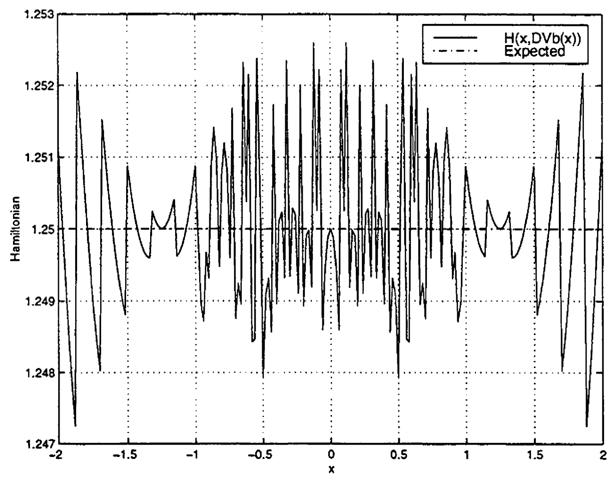


Figure 5.36: Verification that $V_b(x)$ solve the PDE (2.179) for Locally Cubic System (5.65)

5.7 A 2-dimensional Circular Limit Cycle System

5.7.1 System Dynamics

Consider the system Σ expressed in polar coordinates as

$$\Sigma \ : \ \begin{cases} \dot{r} &= r(1-r^2)+v, \\ \dot{\theta} &= 1, \\ z &= r, \end{cases} \tag{5.70}$$

where $r(t)$ and $\theta(t)$ are the radius and angle of the trajectory in $\mathbf{R}^2$ respectively. Defining the change of coordinates

$$\begin{aligned} x_1 &= r\cos\theta, \\ x_2 &= r\sin\theta, \end{aligned}$$

system Σ may be expressed as a limit cycle system in $\mathbf{R}^2$ as

$$\Sigma \ : \ \begin{cases} \dot{x} &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x - |x|^2 x + \frac{x}{|x|}v, \quad |x| \neq 0 \\ z &= x, \end{cases} \tag{5.71}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Clearly Σ expressed as (5.71) is a two dimensional system in $\mathbf{R}^2$. However, by inspection of (5.70), Σ is radially symmetric, since the angle θ does not enter the dynamics of the radius r. That is, when Σ is expressed in polar coordinates, the dynamics are essentially one dimensional. This proves to be a very useful 2-dimensional test for the computation of the storage functions, since the computations can be performed using the 2-dimensional implementation and compared with a 1-dimensional analytical calculation (and also a 1-dimensional implementation).

By inspection of system (5.70), in the absence of disturbances, the radius $r(t)$ tends asymptotically to 1. That is, the system (5.71) exhibits a unit radius limit cycle in the absence of disturbances, as shown in Figure 5.37.

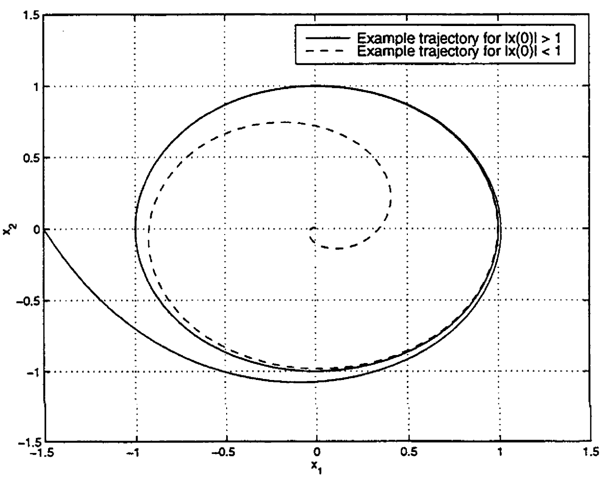


Figure 5.37: Behaviour of System (5.71) in the Absense of Disturbances

5.7.2 The Power Gain Property

With a view to applying Theorem 2.4.8 to demonstrate $\mathcal{FP}$-gain for system Σ, the drift and disturbance terms of (5.71) are given by

$$a(x) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x - |x|^2 x, \tag{5.72}$$

$$b(x) = \frac{x}{|x|}, \quad |x| \neq 0. \tag{5.73}$$

Defining $b(0) = [1 \;\; 0]'$, we find that

$$a(x) \cdot x = |x|^2 - |x|^4 \leq -|x|^2 + 1, \tag{5.74}$$

$$|b(x)| = 1, \tag{5.75}$$

for all $x \in \mathbf{R}^2$. Hence, assumptions (A7), (A10), and (A12) hold, with $C_1 = C_2 = 1$, $L_3 = 1$, and $L_5 = 1$. Hence, by Theorem 2.4.8 (choosing $\delta = 2$), system Σ has $\mathcal{FP}$-gain

≤ 1, with power bias $\lambda = 3$. Hence, by Theorem 2.6.4 and Lemma 2.6.10, the available power λ_a^γ (2.84) is bounded above by $\lambda = 3$ for all gains $\gamma \geq 1$. Furthermore, due to the unit radius limit cycle behaviour in the absence of disturbances, Remark 2.6.3 states that the available power λ_a^γ is bounded below by $\|z\|_{\mathcal{FP}} = 1$. Hence, for all $\gamma \geq 1$,

$$1 \;\leq\; \lambda_a^\gamma \;\leq\; 3. \tag{5.76}$$

5.7.3 Treatment as a 1-dimensional System

Since the 2-dimensional system Σ (5.71) may be expressed as a 1-dimensional system (5.70), the available power λ_a^γ (2.84) may be computed explicitly by applying Theorem 2.10.15, noting that the PDE (2.179) is scalar. So, writing the PDE (2.179) (using the coordinate r),

$$\sup_{v\in\mathbf{R}} \left\{ \nabla_r V(r) \left[r(1-r^2) + v \right] + r^2 - \gamma^2 |v|^2 \right\} \;=\; \lambda.$$

Applying completion of squares, this becomes

$$\frac{1}{4\gamma^2} \left[\nabla_r V(r)\right]^2 + r(1-r^2)\nabla_r V(r) + r^2 \;=\; \lambda. \tag{5.77}$$

Following the same procedure as in preceding scalar examples, we note that (5.77) is quadratic in the gradient $\nabla_r V(r)$. with discriminant

$$\Delta(r) \;=\; r^6 - 2r^4 + \left(1 - \frac{1}{\gamma^2}\right) r^2 + \frac{1}{\gamma^2}\lambda, \tag{5.78}$$

which is clearly cubic in r^2. Differentiating and setting equal to zero yields that $\Delta(r)$ has a local minimum at $\bar{r}$, where

$$\bar{r}^2 \;=\; \frac{2}{3}\left(1 + \sqrt{1 - \frac{3}{4}(1-\gamma^2)}\right).$$

(Note that the RHS is clearly positive for every γ.) For the gradient $\nabla_r V(r)$ to exist, the discriminant must be nonnegative for every $r \geq 0$. That is, $\Delta(\bar{r}) = 0$, which implies from (5.78) that

$$\lambda \;\geq\; \frac{2}{3} - \frac{2}{27}\gamma^2 + \frac{2}{27}(\gamma^2+3)\sqrt{\frac{\gamma^2+3}{\gamma^2}} \;=:\; \lambda^*.$$

By solving quadratic (5.77) for the gradient of the stabilizing solution $\nabla_r V_-(r)$ and integrating, we find that the PDE (2.179) admits infinitely many stabilizing solution pairs. Figure 5.38 illustrates some of these solutions pairs for gain $\gamma = 0.4$, where the number annotating each curve is the offset $\lambda - \lambda^*$.

The important feature of Figure 5.38 is the lack of differentiability of the stabilizing

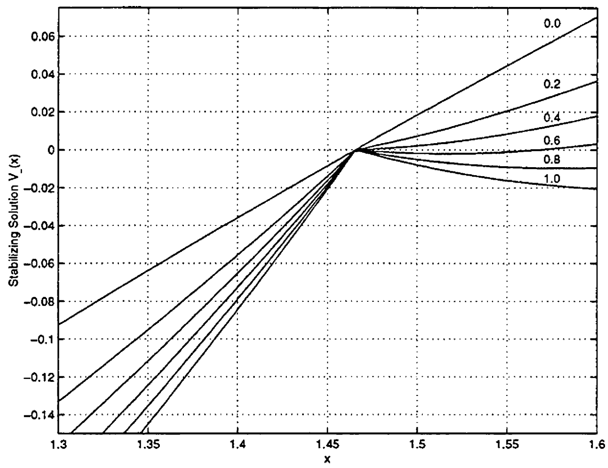


Figure 5.38: Stabilizing Solutions $V_{-}(x)$ corresponding to power bias λ for $\lambda - \lambda^* \in [0, 1]$

solution V_{-} for $\lambda > \lambda^*$. Hence, by Theorem 2.10.15, we conclude that the available power must be λ^*. That is,

$$\lambda_a^\gamma = \frac{2}{3} - \frac{2}{27}\gamma^2 + \frac{2}{27}\left(\gamma^2 + 3\right)\sqrt{\frac{\gamma^2 + 3}{\gamma^2}}. \tag{5.79}$$

Note that, as expected from Lemma 2.6.10, the available power λ_a^γ is a nonincreasing function of the gain γ. This is illustrated in Figure 5.39. Furthermore, note that $\lambda_a^{\gamma=1} = \frac{32}{27} \in [1, 3]$, as expected from (5.76).

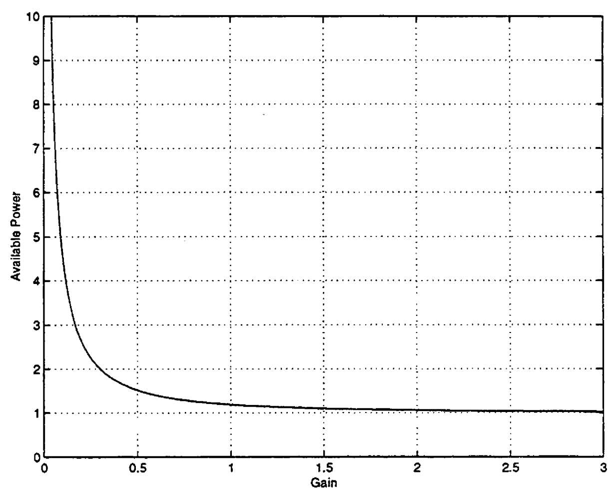


Figure 5.39: Available Power λ_a^γ (5.79) for the 2D Circular Limit Cycle System

We now consider in detail the case where $\gamma = 0.4$.

From (5.79), $\lambda_a^{\gamma=0.4} \approx 1.6951$. By applying Theorem 2.6.11 and (5.79), the power of the worst case disturbance is

$$\begin{aligned} \|v_\gamma^*\|_{\mathcal{FP}} &= \sqrt{\frac{2}{27} + \frac{9 - 3\gamma^2 - 2\gamma^4}{27\gamma^3\sqrt{\gamma^2+3}}} \\ &\approx 1.6826 \end{aligned}$$

for $\gamma = 0.4$. Furthermore, by (numerical) integration of the solutions of (5.77), the stabilizing and antistabilizing solutions can be computed, as illustrated in Figure 5.40.

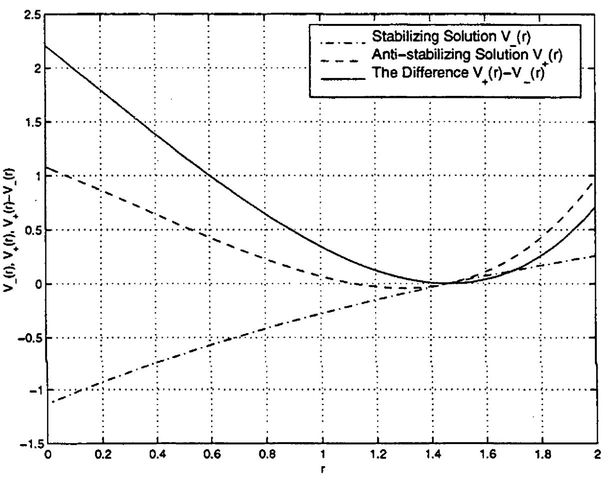


Figure 5.40: Stabilizing and Anti-stabilizing Solutions of the PDE (5.77)

The system (5.70) has a stable equilibrium at r^* in the presence of the worst case disturbance, where $\Delta(r^*) = 0$. This yields that

$$\begin{aligned} r^* &= \sqrt{\frac{2}{3} + \frac{1}{3}\sqrt{1 + \frac{3}{\gamma^2}}} \\ &\approx 1.4656 \end{aligned} \tag{5.80}$$

for $\gamma = 0.4$. Note that $\Delta(r^*) = 0$ corresponds exactly to the case where the gradients of the stabilizing and antistabilizing solutions are equal. That is, r^* is the minimum point of $V_+(r) - V_-(r)$, as in Figure 5.40.

By simulating the 1-dimensional system (5.70) (using SIMULINK$_{\text{TM}}$) in the presence of the worst case disturbance (2.188), the worst case disturbance power, available power, and equilibrium can be computed numerically and compared with the above expected values. Using the model of Figure 5.41, the results of these simulations are illustrated in Figure 5.42.

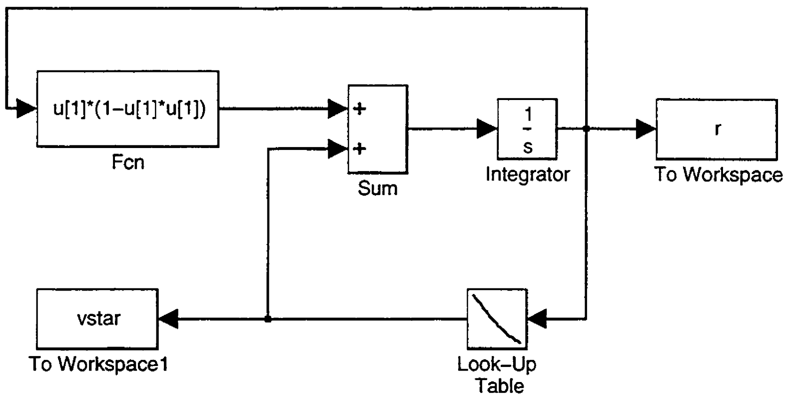


Figure 5.41: SIMULINK$_{\text{TM}}$ Model for the System (5.70) with Worst Case Disturbance

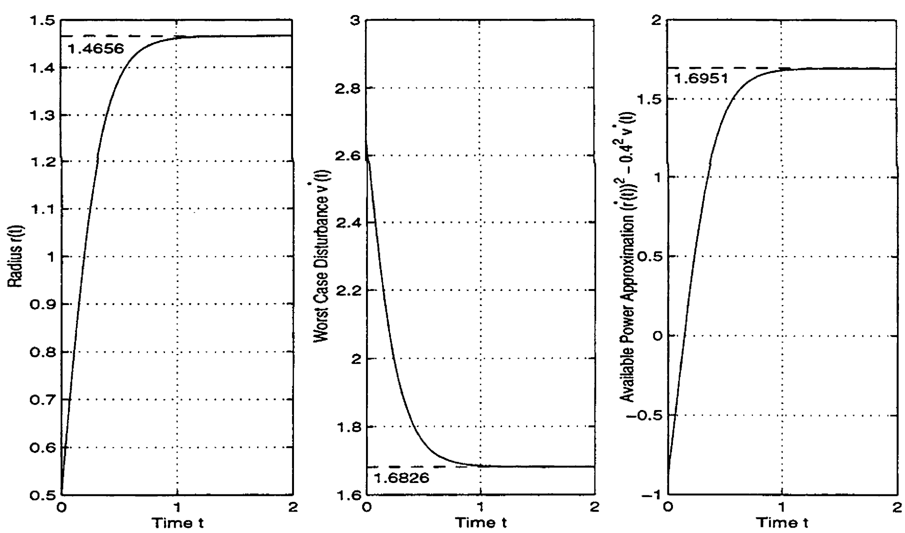


Figure 5.42: Simulation Results for System (5.70) with Worst Case Disturbance

Since the expected results are (for gain $\gamma = 0.4$)

$$\begin{aligned} r^* &\approx 1.4656, \\ \|v^*\|_{\mathcal{FP}} &\approx 1.6826, \\ \lambda_a &\approx 1.6951, \end{aligned}$$

Figure 5.42 indicates that simulation using the 1-dimensional system (5.70) agrees with the 1-dimensional analytical calculations detailed so far in this section.

Finite difference approximations can also be utilized to compute the available power

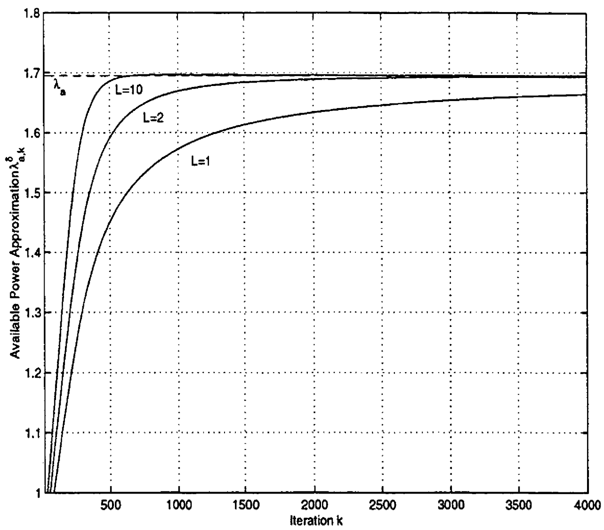


Figure 5.43: Evolution of the Available Power Approximation $\lambda^{\delta}_{a,k}$

/ infinite horizon available storage pair (λ_a, V_b) as in Chapter 4. In this case, we apply a 1-dimensional centered finite difference method, with state space and disturbance coordinate grids of

$$
\begin{aligned}
G_X &= [0.0, 2.0] \cap (\mathbf{R})^{0.01}, \\
G_V &= [1.0, 3.5] \cap (\mathbf{R})^{0.1},
\end{aligned}
$$

respectively. We choose to use the minimal interpolation interval $\delta_T = \frac{\delta_X}{\max_{x \in G_X}\{m(x)\}}$, and the reference state $r_0 = 1.0$ for the centering. After 4000 iterations, the finite difference approximation of $V_b(r) - V_b(r_0)$ has converged to within a maximum relative error on G_X of 10^{-15}. Figure 5.44 illustrates the comparison between this approximation and the stabilizing PDE solution $V_-(r)$ of Figure 5.40. Note that $V_b(r) - V_b(r_0) - V_-(r) \in [0.2772, 2797]$ for all $r \in G_X$. Figure 5.45(a) illustrates the corresponding worst case disturbance for $V_b(r)$, which is clearly a very coarse approximation. However, reducing δ_V as in Figure 5.45(b) has little effect on the variability of $V_b(r) - V_b(r_0) - V_-(r)$. Alternatively, reducing the state space discretization δ_X results in a promising reduction, as shown in Figure 5.44(b). This decrease in variability with decreasing δ_X supports the conjecture that the stabilizing solution $V_-(r)$ is, to within an additive constant, equal to the infinite horizon available storage $V_b(r)$.

In addition to the computation of the finite difference approximation of the infinite horizon available storage $V_b(r)$, the centered scheme also computes an approximation to the available power λ_a. The evolution of this approximation for a number of acceleration factors ($L = 1$ implies no acceleration) is illustrated in Figure 5.43.

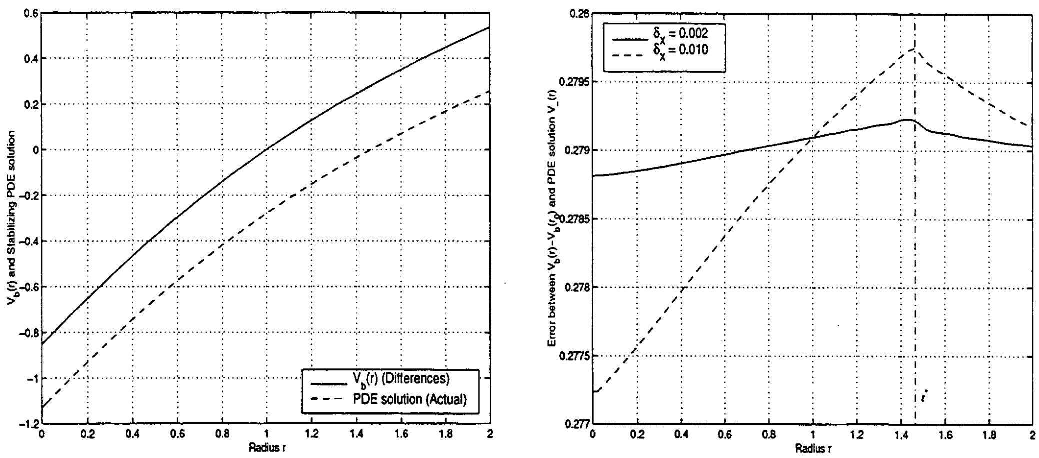
Using another centered finite difference scheme from Chapter 4, the infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$ may also be calculated. In this case, we choose state space and disturbance coordinate grids

$$\begin{aligned} G_X &= [0.0, 2.0] \cap (\mathbf{R})^{0.01}, \\ G_V &= [-3.5, 10.5] \cap (\mathbf{R})^{0.5}, \end{aligned}$$

respectively. We again use the minimal interpolation interval, and the reference state $r_0 = 1.0$ for centering. Since it is the infinite horizon fixed initial state required supply to be calculated, we must choose the initial state ξ. Having found that $r = 0$ minimizes the infinite horizon available storage $V_b(r)$, the natural choice of initial state is $\xi = 0$. This is implemented in the algorithm by setting the initial data to be

$$V_{br,k=0}^{f,\delta}(\xi, r) = \begin{cases} 0 & r = \xi, \\ \infty & \text{elsewhere.} \end{cases}$$

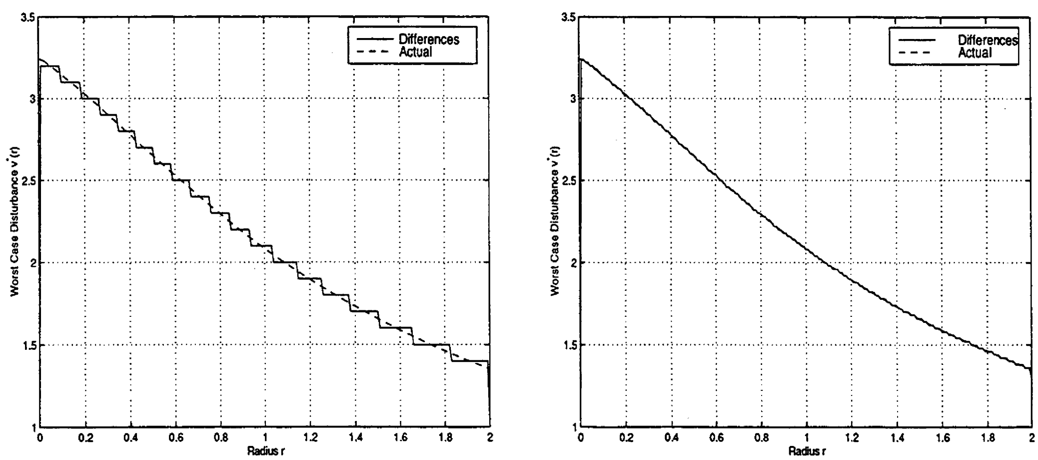
After 10000 iterations, the finite difference approximation of $V_{br}^f(\xi, r) - V_{br}^f(\xi, r_0)$ has converged to within a maximum relative error on G_X of order 10^{-16}. Comparisons with the antistabilizing PDE solution V_+ are drawn in Figures 5.46 and 5.47.



(a) Finite Difference Approximation for $V_b(r) - V_b(r_0)$

(b) Error between $V_b(r) - V_b(r_0)$ Approximation and the Stabilizing PDE Solution

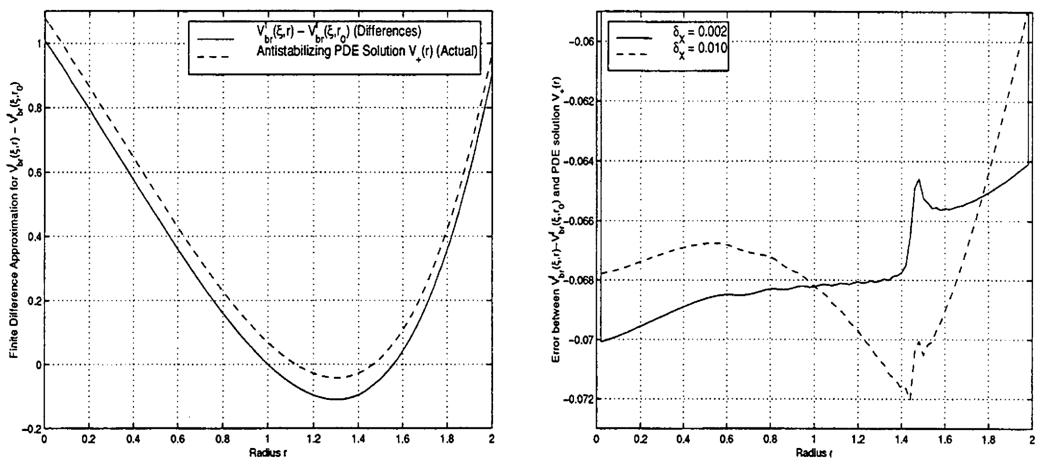
Figure 5.44: Finite Difference Approximation for $V_b(r) - V_b(r_0)$



(a) $\delta_V = 0.1$

(b) $\delta_V = 0.02$

Figure 5.45: Finite Difference Approximation for the Worst Case Disturbance for $V_b(r)$



(a) Finite Difference Approximation for $V^f_{br}(\xi, r) - V^f_{br}(\xi, r_0)$

(b) Error between $V^f_{br}(\xi, r) - V^f_{br}(\xi, r_0)$ Approximation and the Antistabilizing PDE Solution

Figure 5.46: Finite Difference Approximation for $V^f_{br}(\xi, r) - V^f_{br}(\xi, r_0)$

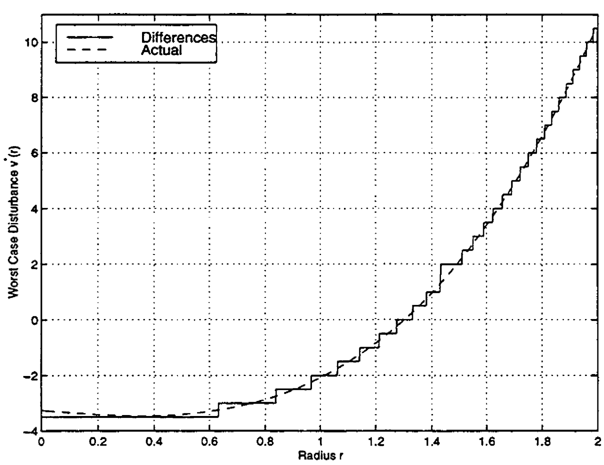


Figure 5.47: Finite Difference Approximation for the Worst Case Disturbance for $V^f_{br}(\xi, r)$

5.7.4 Treatment as a 2-dimensional System

So far, system (5.70) has been treated as a 1-dimensional system with the radius of the trajectory as the state. In this section, we focus on the 2-dimensional form of the state equation (5.71), where the state is in Euclidean coordinates $\mathbf{R}^2$. This is motivated by the need to verify the finite difference approximation for $V_b(x)$ and $V_{br}^f(\xi, x)$ when computed as functions on $\mathbf{R}^2$, using the 1-dimensional results already presented.

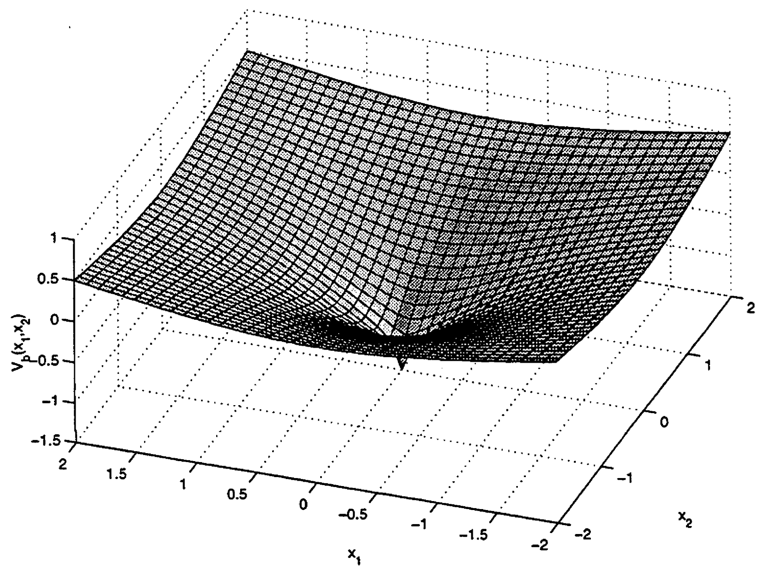


Figure 5.48: Finite Difference Approximation for $V_b(x) - V_b(x_0)$

We apply a 2-dimensional centered finite difference approximation method, with state space and disturbance coordinate grids of

$$\begin{aligned} G_X &= \left\{x \in \mathbf{R}^2 : |x| \leq 2\right\} \cap (\mathbf{R}^2)^{0.05}, \\ G_V &= [0, 4] \cap (\mathbf{R})^{0.2}, \end{aligned}$$

respectively. Again we choose to use the minimal interpolation interval, but now have a reference state $x = [1.6 \quad 0]'$ for the centering. Performing 20000 iterations on a Fujitsu VPP300 supercomputer yields a final relative error of order 10^{-5} for the finite difference approximation $V_b(x) - V_b(x_0)$, which is illustrated along with the worst case disturbance in Figures 5.48 and 5.49.

Similarly, a 2-dimensional centered finite difference method can be employed to compute an approximation for $V_{br}^f(\xi, x) - V_{br}^f(\xi, x_0)$. Since V_b has its minimum at the origin, it is natural to choose $\xi = 0 \in \mathbf{R}^2$. It is again convenient to choose a reference

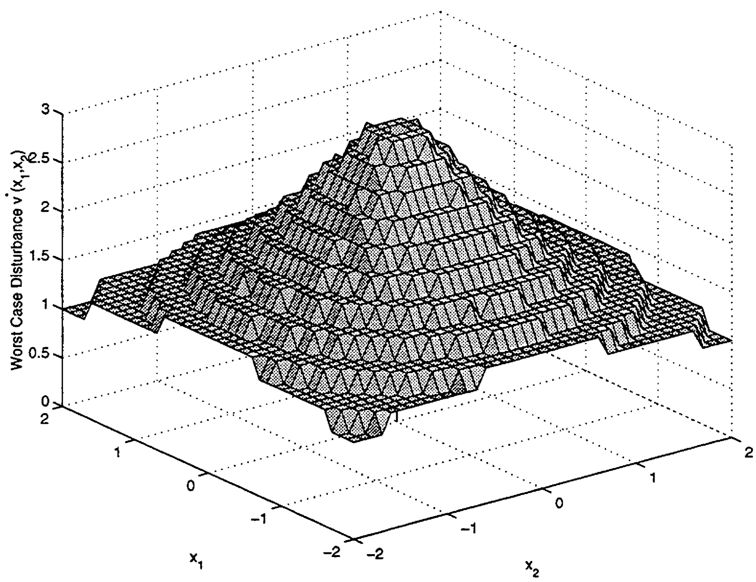


Figure 5.49: Finite Difference Approximation for the Worst Case Disturbance for $V_b(x) - V_b(x_0)$

state $x_0 = [1.6 \;\; 0]'$ for the centering. Finally, the state space and disturbance coordinate grids are chosen to be

$$\begin{aligned} G_X &= \left\{x \in \mathbf{R}^2 : |x| \leq 2\right\} \cap (\mathbf{R}^2)^{0.05}, \\ G_V &= [-4.0, 12.0] \cap (\mathbf{R})^{1.0}. \end{aligned}$$

After 20000 iterations, the relative error is of order 10^{-3}, with an absolute error of order 10^{-5}. The resulting approximation for $V_{br}^f(\xi, x) - V_{br}^f(\xi, x_0)$ and corresponding worst case disturbance is illustrated in Figures 5.50 and 5.51.

With approximations for the (x_0 zeroed) infinite horizon fixed initial state required supply $V_{br}^f(\xi, x) - V_{br}^f(\xi, x_0)$ and the (x_0 zeroed) infinite horizon available storage $V_b(x) - V_b(x_0)$, it is now a simple matter of subtraction to compute an approximation for the (x_0 zeroed) $W_\xi(x) - W_\xi(x_0)$ function, and hence determine the behaviour of the system in the presence of the worst case disturbance. As illustrated in Figures 5.52 and 5.53, the worst case dynamics appear to lie on a circle of radius approximately equal to $r^* \approx 1.4656$, as calculated in Section 5.7.3.

By taking any radial cross-section of Figures 5.48 and 5.50, it is immediately possible to compare the 2-dimensional finite difference computation with the 1-dimensional computations of Section 5.7.3. For example, choosing the cross-section corresponding to $x_2 = 0$ yields the comparison shown in Figure 5.54.

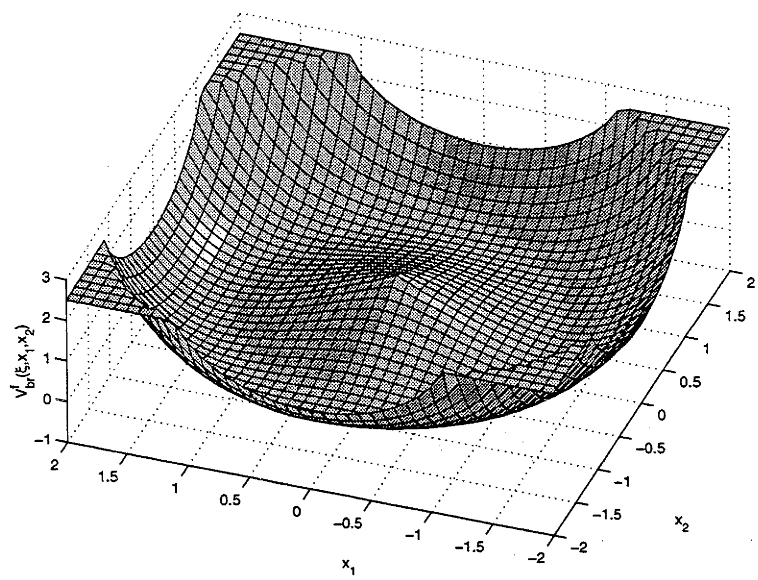


Figure 5.50: Finite Difference Approximation for $V^f_{br}(\xi, x) - V^f_{br}(\xi, x_0)$

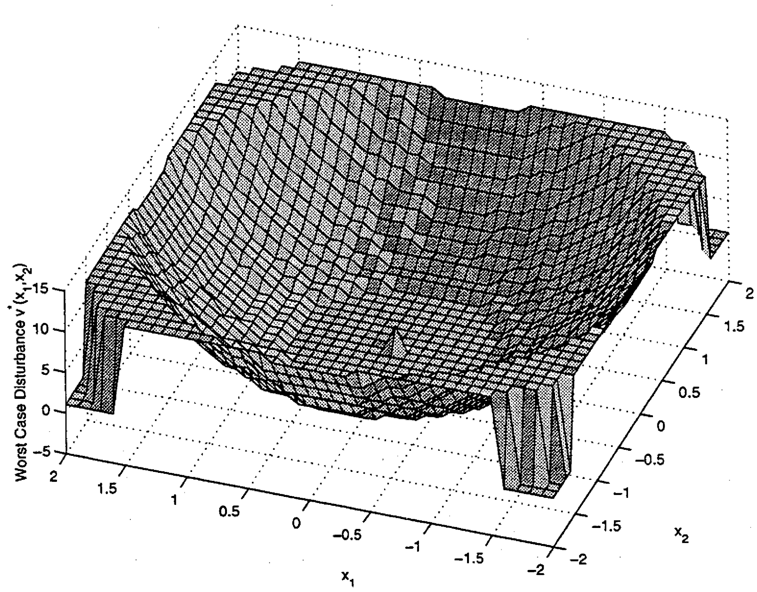


Figure 5.51: Finite Difference Approximation for the Worst Case Disturbance for $V^f_{br}(\xi, x) - V^f_{br}(\xi, x_0)$

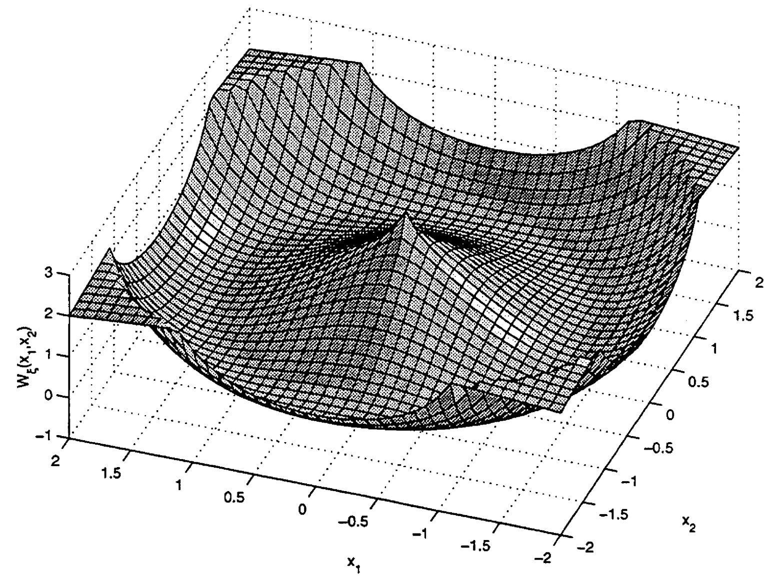


Figure 5.52: Finite Difference Approximation for $W_\xi(x) - W_\xi(x_0)$

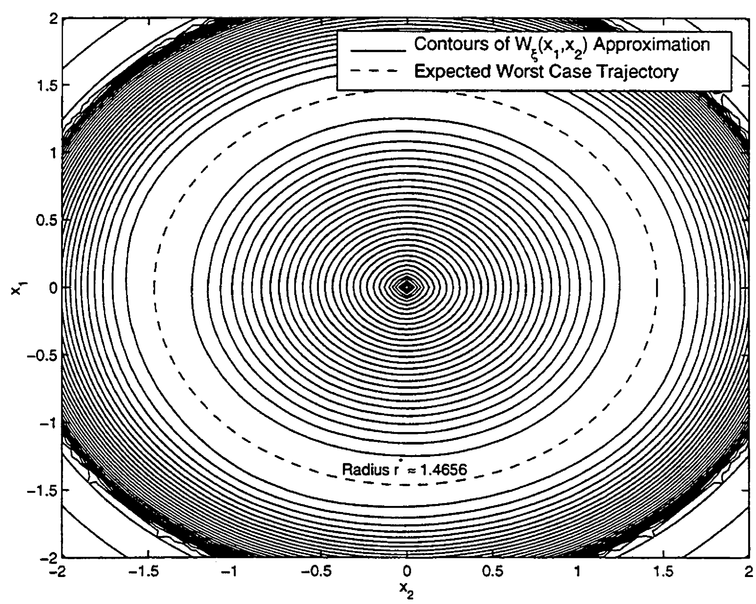


Figure 5.53: Contours of the $W_\xi(x) - W_\xi(x_0)$ Approximation and the Expected Worst Case Trajectory

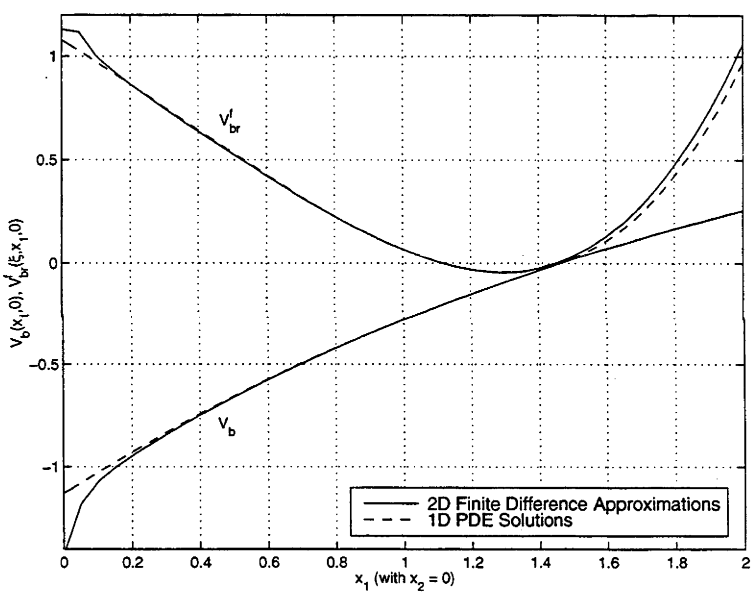


Figure 5.54: A Normalized Comparison between 2-dimensional Finite Difference Computations and the 1-dimensional PDE Solutions

5.8 A 2-dimensional Non-circular Limit Cycle System

The circular limit cycle system of Section 5.7 is a very useful example since it illustrates the transition from the analysis of one dimensional systems to the analysis of two dimensional systems. The fact that the system (5.70) could be expressed as a one dimensional system allowed the various value functions computed on $\mathbf{R}^2$ to be compared with the corresponding value functions computed on $\mathbf{R}$.

A direct consequence of this equivalence between one dimension and two is angular invariance of value functions. In particular, $V_{br}^f(\xi, x)$, $V_b(x)$, and hence $W(x)$ must all be constructed from rotations of the corresponding one dimensional quantities around the vertical (or z) axis. Hence, any worst case trajectory must be circular; simply a scaled version of the disturbance free trajectory.

In this section, a slight modification of the circular limit cycle system of Section 5.7 is considered. The aim is to move away from one dimensional systems and consider a truly two dimensional limit cycle system which does not have the angular invariance of system (5.70). To do this, consider the modified limit cycle system

$$\Sigma \ : \ \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 - x_1\left(x_1^2 + \mu x_2^2\right) \\ x_1 + x_2 - x_2\left(x_1^2 + x_2^2\right) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v, \\ z = x_1^2 + x_2^2. \end{cases} \tag{5.81}$$

The obvious change from system (5.70) is that system (5.81) cannot exhibit a circular limit cycle in the absence of disturbances (due to the coefficient μ in the $\dot{x}_1$ equation), as shown in Figure 5.55 for $\mu = 5$. Furthermore, the disturbance v no longer enters the system radially.

System Σ may be shown to exhibit power gain applying Theorem 2.4.8. In order to apply this theorem, assumptions (A7), (A10), and (A12) must be shown to hold. Writing the drift term of system Σ (5.81) as

$$a_\mu(x) = \begin{bmatrix} x_1 - x_2 - x_1\left(x_1^2 + \mu x_2^2\right) \\ x_1 + x_2 - x_2\left(x_1^2 + x_2^2\right) \end{bmatrix},$$

then,

$$\begin{aligned} a_\mu(x) \cdot x &= x_1^2 - x_1^2\left(x_1^2 + \mu x_2^2\right) + x_2^2 - x_2^2\left(x_1^2 + x_2^2\right) \\ &= |x|^2\left(1 - |x|^2\right) - (\mu - 1)\, x_1^2 x_2^2 \\ &\le |x|^2\left(1 - |x|^2\right) \end{aligned}$$

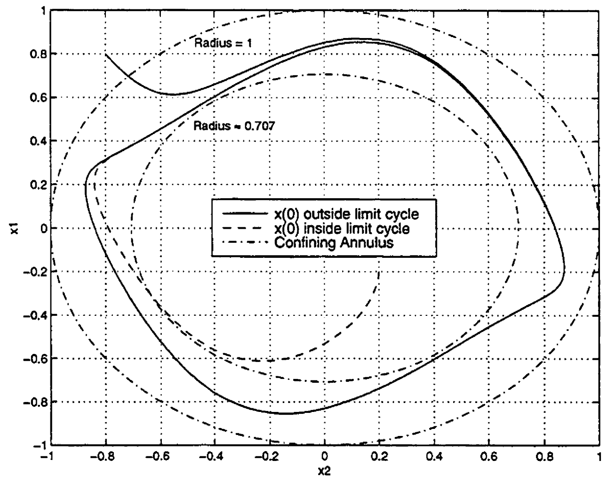


Figure 5.55: Behaviour of System (5.81) ($\mu = 5$) in the Absense of Disturbances and the Confining Annulus (5.82)

for $\mu \geq 1$. However, this is precisely (5.74). Letting $b(x)$ be the disturbance coefficient in (5.81), clearly $|b(x)| = 1$, which is precisely (5.75). Finally, since $c(x) = |x|^2$ as in (5.71), assumptions (A7), (A10), and (A12) hold with the same constants as for the circular limit cycle system. That is, for any $\mu \geq 1$, system Σ (5.81) has $\mathcal{FP}$-gain ≤ 1 with power bias $\lambda = 3$ and energy bias $|x|^2$.

Note that system Σ (5.81) may be expressed in polar coordinates. In particular, noting that $r\dot{r} = a(x) \cdot x$ and considering the disturbance free case,

$$r\,\dot{r} \;=\; r^2 - \frac{1}{2}r^4\,(3 - \cos 4\theta)\,.$$

Hence, $r > 0$ implies that for all θ,

$$r - 2r^3 \;\leq\; \dot{r} \;\leq\; r - r^3.$$

Consequently, for all θ,

$$\begin{aligned} r \leq \frac{1}{\sqrt{2}} \;&\Rightarrow\; \dot{r} \geq 0, \\ r \geq 1 \;&\Rightarrow\; \dot{r} \leq 0. \end{aligned}$$

That is, the disturbance free limit cycle is contained in the annulus

$$A(r) \;=\; \left\{ |r| \in \left[\frac{1}{\sqrt{2}}, 1\right] \right\}, \tag{5.82}$$

as shown in Figure 5.55. Applying Remark 2.6.3 implies that the available power $\lambda_a \geq \frac{1}{2}$. Hence, combining the bounds for λ_a, the available power λ_a (2.84) for system

Σ (5.81) ($\mu = 5$) satisfies for all $\gamma \geq 1$,

$$\frac{1}{2} \leq \lambda_a^\gamma \leq 3. \tag{5.83}$$

(Compare with (5.76) for the circular limit cycle system ($\mu = 1$).)

As with the circular limit cycle system, the infinite horizon available storage $V_b(x)$ (2.169), the infinite horizon fixed initial state required supply (2.199), and the function $W(x)$ (2.211) may be computed numerically using the finite difference approximations of Chapter 4.

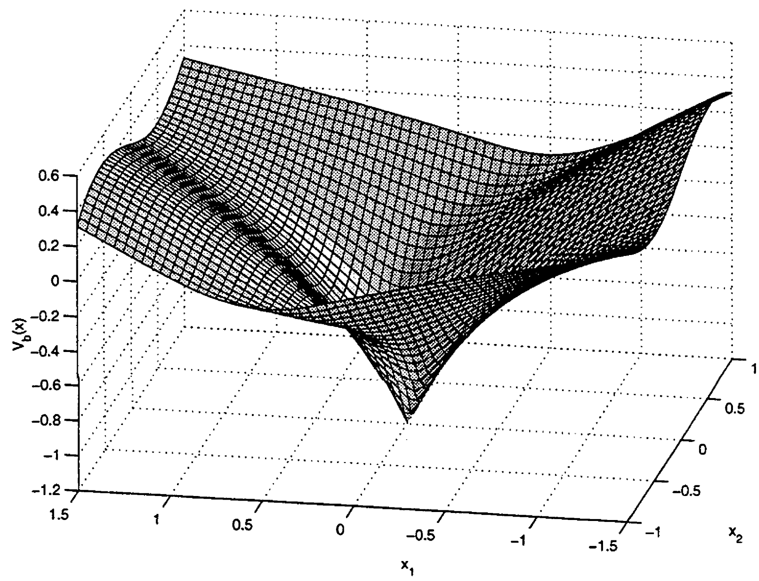


Figure 5.56: Infinite Horizon Available Storage $V_b(x)$ (2.169) for the Non-circular Limit Cycle System (5.81) ($\mu = 5$)

Comparing Figures 5.48 and 5.56 for example, it is clear that the angular invariance of the storage functions present in the circular limit cycle system (5.70) is not present in the modified limit cycle system (5.81).

It is important to note that the infinite horizon fixed initial state required supply $V_{br}^f(\xi, x)$ grows very quickly with $|x|$ for the non-circular limit cycle system (5.81). The flat boundary regions in Figures 5.57 and 5.58 represent truncations, necessary due to growth rates of order $|x|^4$. With these growth rates in mind, Figure 5.59 illustrates the relationship between the minimum of $W(x)$ (2.211) and the worst case trajectory. Although the approximation of $W(x)$ shown in Figure 5.59 is not constant around the worst case trajectory, the maximum absolute variation of $W(x)$ on this trajectory is of

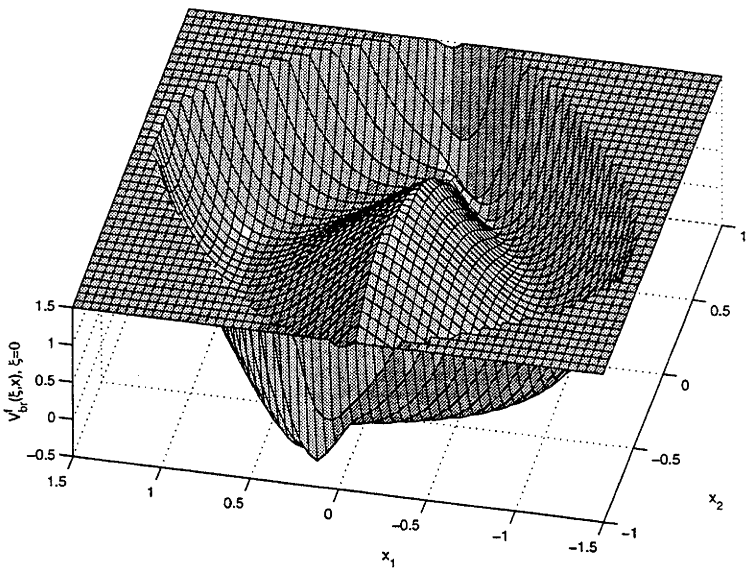


Figure 5.57: Infinite Horizon Fixed Initial State Required Supply $V^f_{br}(\xi, x)$ (2.199) with $\xi = 0$, for the Non-circular Limit Cycle System (5.81) ($\mu = 5$)

order 10^{-1} (on an approximation which grows to order 10^3 within $|x| \leq 2$).

Note that since system (5.81) is 2-dimensional (with no 1-dimensional representation), the available power is no longer trivial to calculate as an explicit function of the gain γ. However, the centered finite difference approximation method of Section 4.3 computes an approximation for the infinite horizon available storage and the available power. This approximation (as a function of the iteration number) is illustrated in Figure 5.60.

Using the approximation $\lambda_a \approx 0.75$, the finite difference method of Section 4.7 may be applied to compute an approximation for the super available storage $V_a(x)$, Figure 5.61. In order to verify Theorem 2.7.4 for storage function V_a, choose $\delta = 0.01$. Then, defining M_δ as in Theorem 2.7.4 and computing M'_δ (2.144), clearly the unforced trajectory of Figure 5.62 tends to the set $M'_\delta \supseteq M_\delta \supseteq D$, where D is the zero set of $V_a(x)$. (Following the proof of Theorem 2.7.4, the maximum excursion from ∂M_δ can be computed by repeated simulation to be 0.0574. Hence, $M_\delta = \{x \in \mathbf{R}^2 : |x|^2 \leq 0.76\}$, $M'_\delta = \{x \in \mathbf{R}^2 : |x|^2 \leq 0.8174\}$. Note that alternatively, $B \approx 0.2546$ and $\sup_{x \in \partial M_\delta} \sup_{s \in (0, \frac{B}{\delta}]} \{|a(x(s))| : x(0) = x\} \approx 0.0574$, so that $L \leq 1.46$. This would yield $M'_\delta = \{x \in \mathbf{R}^2 : |x|^2 \leq 0.76 + 1.46\}$.)

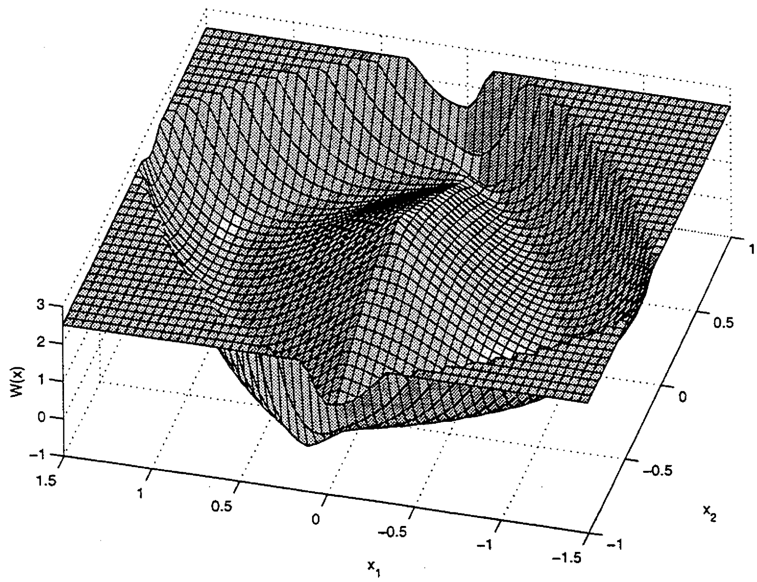


Figure 5.58: $W(x)$ (2.211) for the Non-circular Limit Cycle System (5.81) ($\mu = 5$)

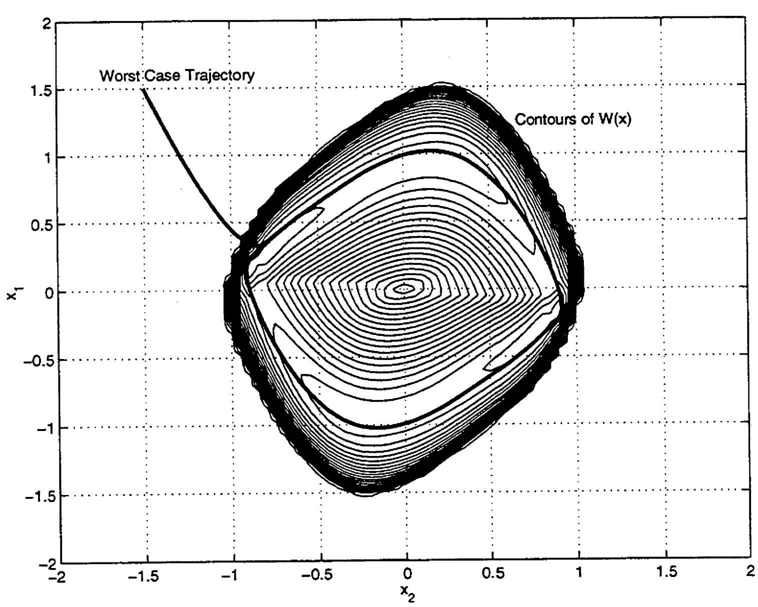


Figure 5.59: The Worst Case Trajectory and the Contours of $W(x)$ (2.211) for the Non-circular Limit Cycle System (5.81) ($\mu = 5$)

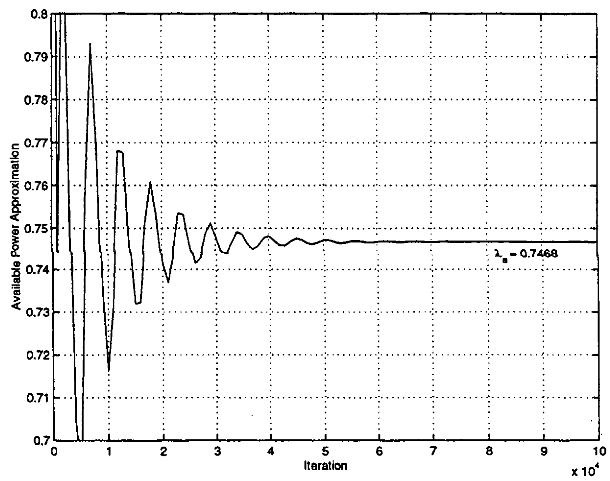


Figure 5.60: Available Power Approximation $\lambda^{\delta}_{a,k}$ for the Non-circular Limit Cycle System (5.81) ($\mu = 5$)

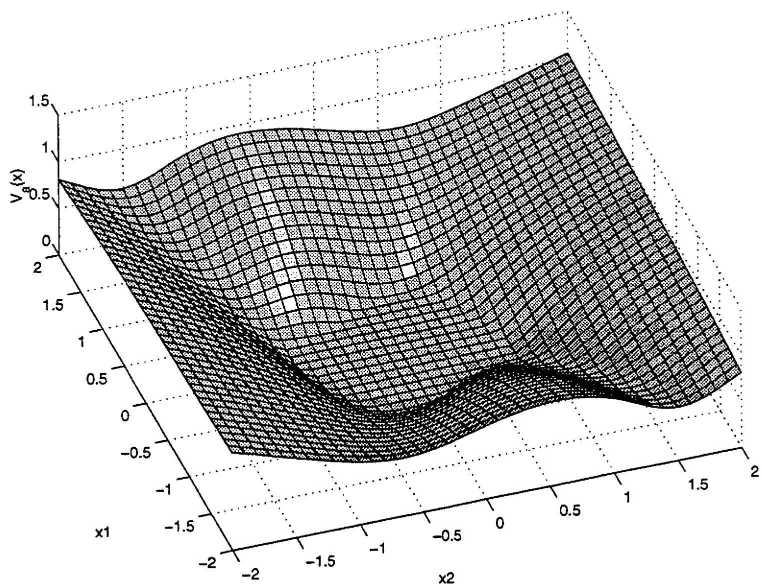


Figure 5.61: Super Available Storage Approximation $V^{\delta}_{a,k}$ for the Non-circular Limit Cycle System (5.81) ($\mu = 5$)

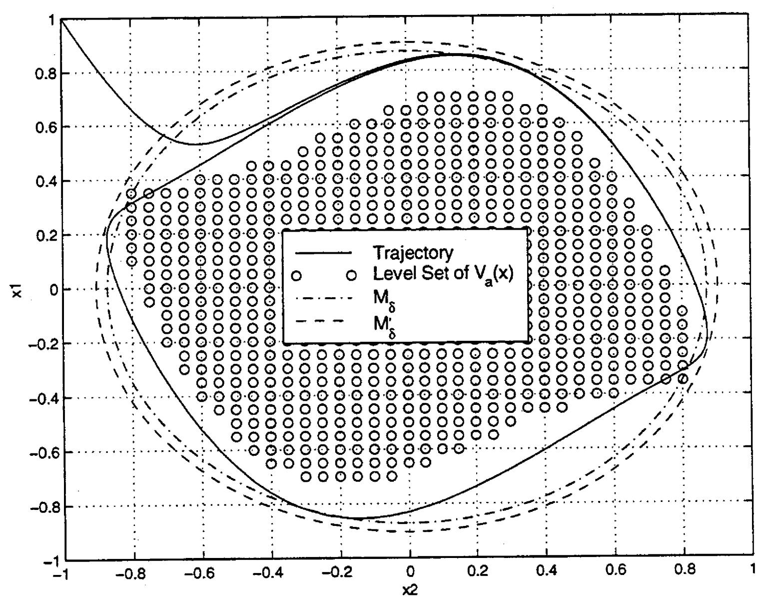


Figure 5.62: An Unforced Trajectory of the Non-circular Limit Cycle System (5.81) ($\mu = 5$) and the Zero Set of $V_a(x)$

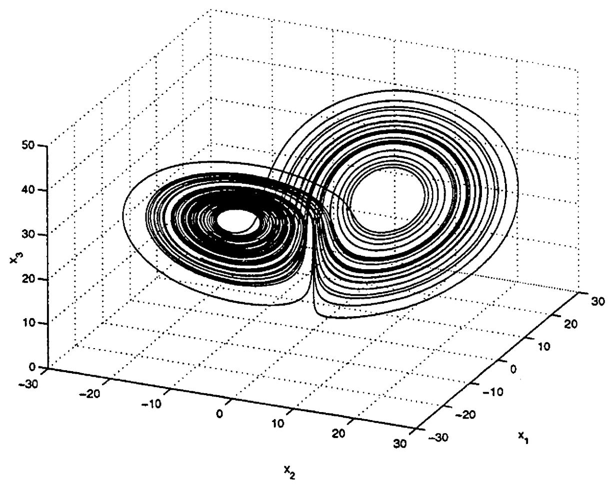


Figure 5.63: Dynamics for the Unforced Lorentz Attractor (5.84) ($v = 0$)

5.9 The Lorentz Attractor

The stability of some limit cycle systems may be analysed using linearization techniques [5, 18]. However, since these techniques ultimately involve approximating the dynamical behaviour in a neighbourhood of the steady state dynamics, such techniques cannot be applied to chaotic systems. Application of power gain analysis techniques on the other hand do not require the use of linearizations. Instead, the only requirement is the existence of an attracting set for the dynamics. An example of such a system is a forced Lorentz attractor,

$$\Sigma \; : \; \left\{ \begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} & = & \begin{bmatrix} 10(x_2 - x_1) \\ -x_2 + 28x_1 - x_1x_3 \\ -\frac{8}{3}x_3 + x_1x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v, \\ z & = & \sqrt{x_1^2 + x_2^2 + x_3^2}. \end{array} \right. \tag{5.84}$$

The unforced dynamics of system (5.84) is illustrated in Figure 5.63.

In this section, we demonstrate that the super available storage (2.149) and the infinite horizon available storage (2.169) can be computed for a chaotic system (in particular, (5.84)).

The super available storage is illustrated in Figures 5.64 through 5.71, with each figure representing a different value of the third state variable x_3. Similarly, the infinite horizon available storage is illustrated in Figures 5.72 through 5.79.

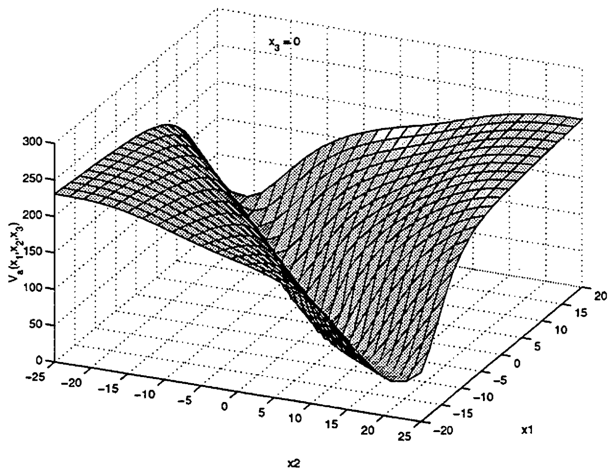


Figure 5.64: The Super Available Storage $V_a(x_1, x_2, x_3)$ for $x_3 = 0$

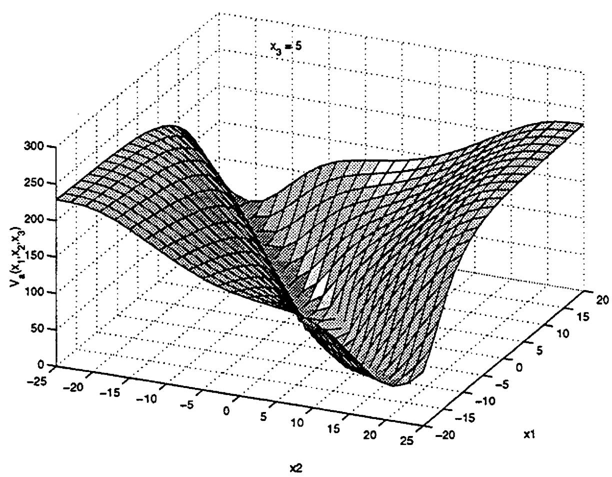


Figure 5.65: The Super Available Storage $V_a(x_1, x_2, x_3)$ for $x_3 = 5$

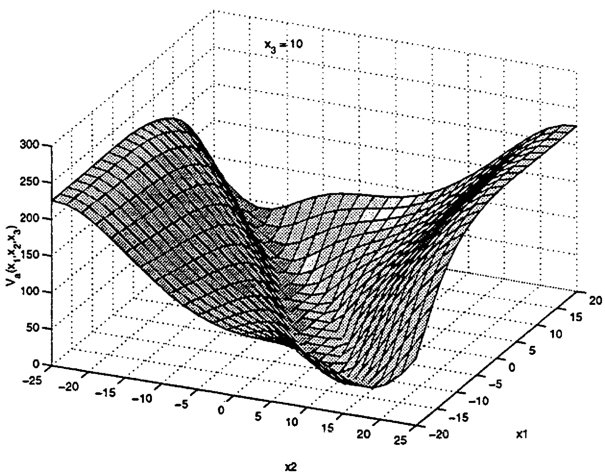


Figure 5.66: The Super Available Storage $V_a(x_1, x_2, x_3)$ for $x_3 = 10$

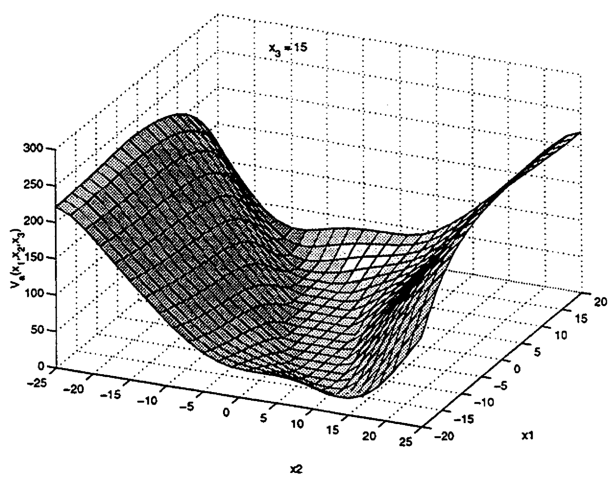


Figure 5.67: The Super Available Storage $V_a(x_1, x_2, x_3)$ for $x_3 = 15$

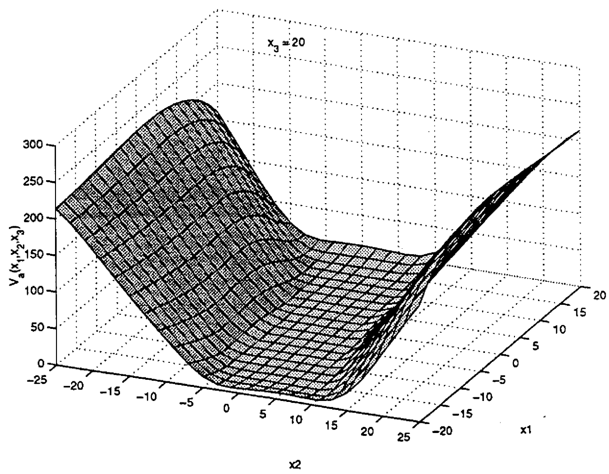


Figure 5.68: The Super Available Storage $V_a(x_1, x_2, x_3)$ for $x_3 = 20$

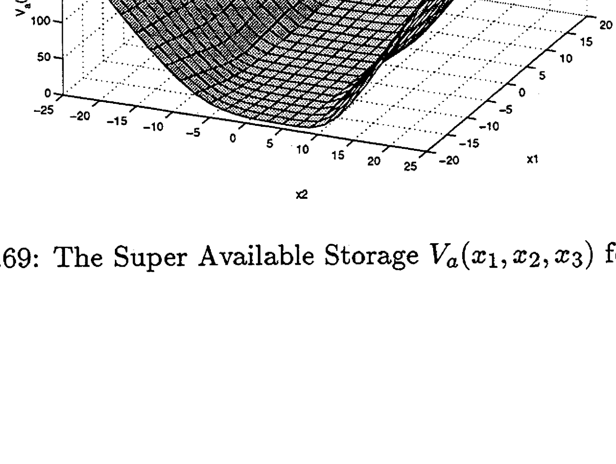


Figure 5.69: The Super Available Storage $V_a(x_1, x_2, x_3)$ for $x_3 = 25$

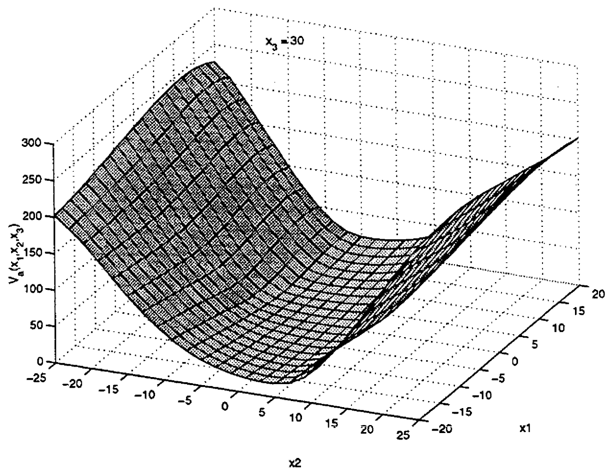


Figure 5.70: The Super Available Storage $V_a(x_1, x_2, x_3)$ for $x_3 = 30$

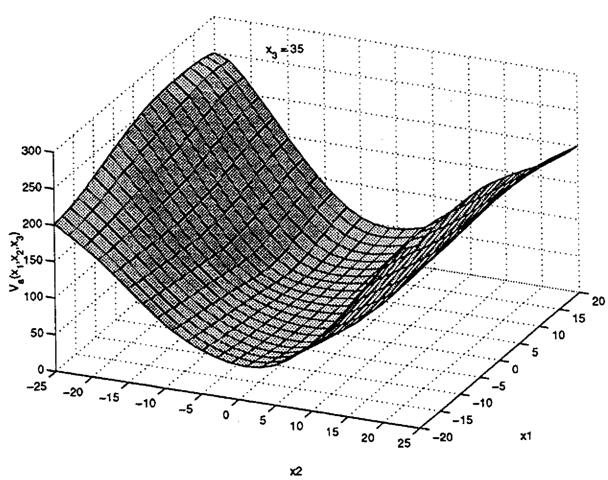


Figure 5.71: The Super Available Storage $V_a(x_1, x_2, x_3)$ for $x_3 = 35$

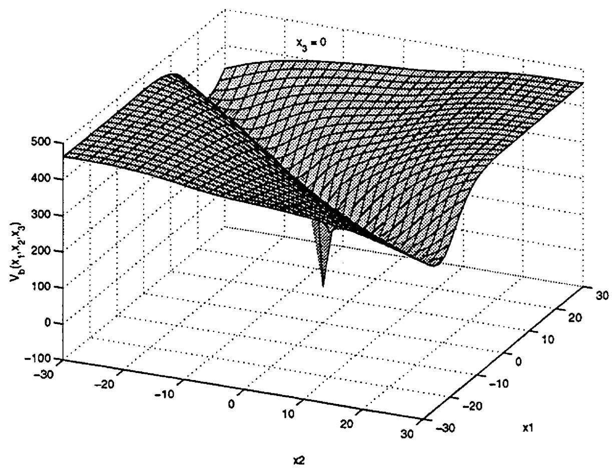


Figure 5.72: The Infinite Horizon Available Storage $V_b(x_1, x_2, x_3)$ for $x_3 = 0$

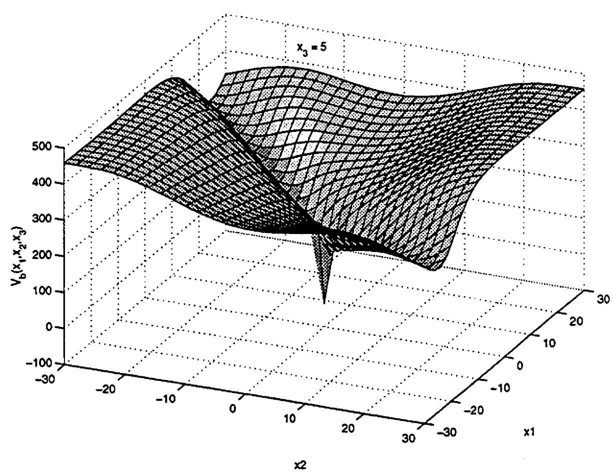


Figure 5.73: The Infinite Horizon Available Storage $V_b(x_1, x_2, x_3)$ for $x_3 = 5$

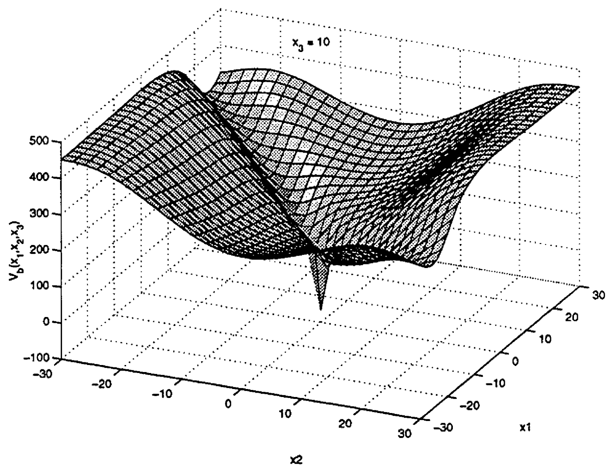


Figure 5.74: The Infinite Horizon Available Storage $V_b(x_1, x_2, x_3)$ for $x_3 = 10$

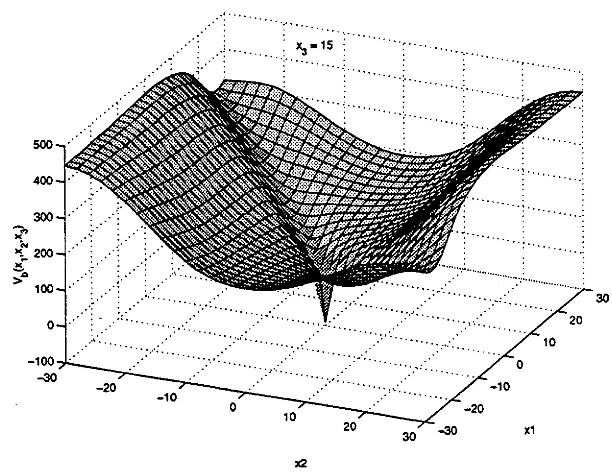


Figure 5.75: The Infinite Horizon Available Storage $V_b(x_1, x_2, x_3)$ for $x_3 = 15$

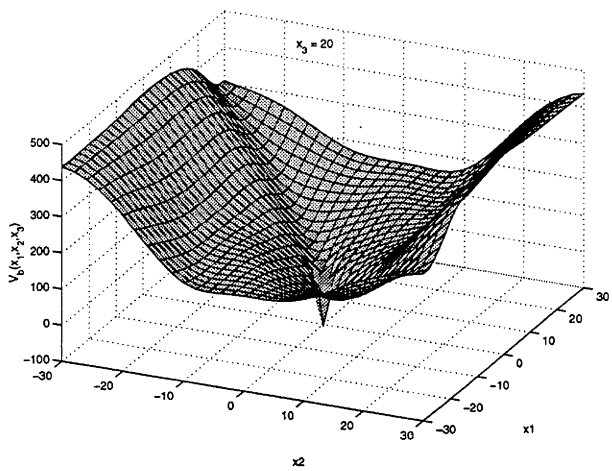


Figure 5.76: The Infinite Horizon Available Storage $V_b(x_1, x_2, x_3)$ for $x_3 = 20$

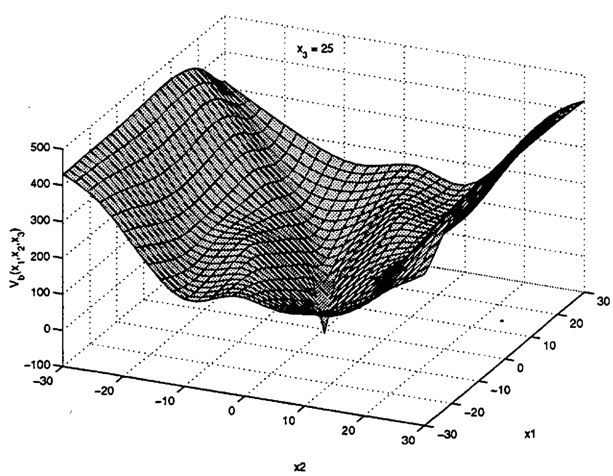


Figure 5.77: The Infinite Horizon Available Storage $V_b(x_1, x_2, x_3)$ for $x_3 = 25$

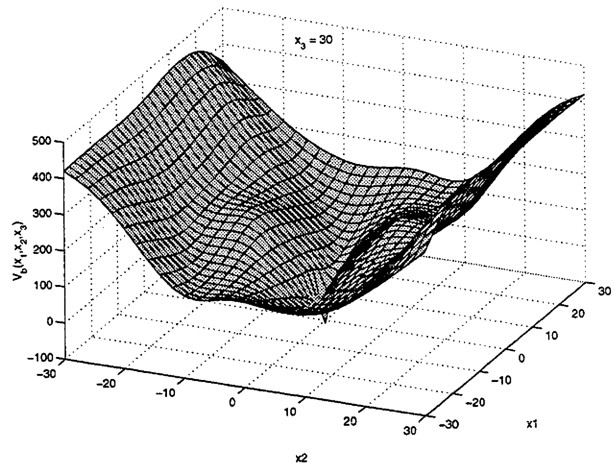


Figure 5.78: The Infinite Horizon Available Storage $V_b(x_1, x_2, x_3)$ for $x_3 = 30$

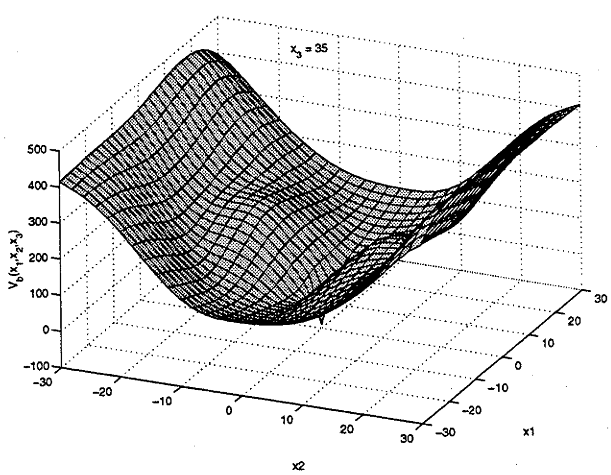


Figure 5.79: The Infinite Horizon Available Storage $V_b(x_1, x_2, x_3)$ for $x_3 = 35$

Chapter 6

Conclusions

6.1 Conclusions

The main theme of this thesis has been the analysis of nonlinear systems which exhibit power gain from disturbance to output. In contrast to $\mathcal{L}_2$-gain analysis techniques, the tools presented in this thesis admit the treatment of nonlinear systems which exhibit some form of internal power generation, such as limit cycles. This power generation was captured by the concept of available power, which was shown to be fundamental in the analysis of systems with power gain. Following the thread of energy dissipative systems theory, it was possible to develop an analogous notion of power dissipativity, which allowed the energy balance analysis which has proved so useful in $\mathcal{L}_2$-gain analysis to be applied to the power gain case. Corresponding definitions of available storage and required supply have been proposed, with the internal power generation of the system accounted for using the notion of available power. Analysis of nonlinear systems with power gain in the presence of the worst case disturbance has revealed a new wealth of system behaviour which does not arise in systems with $\mathcal{L}_2$-gain. Unlike the $\mathcal{L}_2$-gain case, in which equlibria are preserved in the presence of the worst case disturbance, the power gain case gives rise to fundamental changes in dynamics, such as shifts in equilibria, bifurcation of equilibria, and the modification of shape and size of limit cycles.

In pursuing an application of power gain analysis in control theory, an optimal power gain control problem was defined. In this problem, the optimal control was defined to be that which minimizes the limit cycle behaviour of a closed loop system.

This was formulated as a minimization over a set of possible control policies of the available power, which is a measure of the internal power generation of the system. The resulting state feedback controller synthesis techniques were then applied to a class of linear systems with actuator nonlinearities.

Finally, a number of numerical techniques for computing the available power, available storage, and required supply were discussed. These techniques were then applied to a range of explicit nonlinear examples.

6.2 Further Work

The development of power gain analysis and control theory is new. As such, there is a great deal of future work to be undertaken.

In particular, a deeper understanding of the issue of stability is required. The connections between the available storage / required supply and stabilizing / antistabilizing solutions of the attendant stationary PDE have been demonstrated in explicit cases but not exploited in general theory. Existence of PDE solutions and characterization of solutions also remains outstanding. Precise connections between the super and infinite horizon available storage also need to be fully understood. Much of the PDE machinery for the required supply remains to be developed or proved.

With the broadening of understanding of power gain analysis techniques, further work on formulating both state feedback and measurement feedback control techniques is essential. Although an optimal state feedback problem is proposed in this thesis, useful numerical techniques for the solution of such problems requires attention. Needless to say, the measurement feedback control problem is a completely new and no doubt complex problem.

Finally, the proof of convergence of numerical schemes is required. Although the numerical techniques presented rely on existing schemes developed for stochastic control problems, the determinism of the power gain problem requires that analogous deterministic proofs be developed.

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